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THE DYNAMIC CHARACTERISTICS OF A
NONLINEARLY DAMPED SYSTEM

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

NOMENCLATURE

C_n	Damping coefficients for $n = 0,1,2,\cdots$.
CF	Criterion function.
CS	Scaled damping coefficients.
C_{ij}, D_{ij}, d_{ij}	Constants resulting in Perturbation Series Approximation--over damped case.
$F(\dot{x}), f(\dot{x})$	Dissipation function.
FACTOR	Convergence factor in search routine.
K	Spring rate.
L,LS,LLS	Interval lengths in Golden Section routine.
m	Mass.
SFACTOR	Factor for scaling damping coefficients.
$Sgn(\dot{x})$	Sign function--Equation (1.1).
t	Time.
x	Displacement.
$\dot{x}$	Velocity.
$\ddot{x}$	Acceleration.
x_a	Approximate response.
x_m	Measured response.
$\hat{x}_m$	Measured response with noise present.
X_o	Initial displacement.
X^*	Amplitude of motion corrected for presence of Coulomb friction.
X(t)	Time varying amplitude of motion.

$y(s)$	Laplace transformation.
α,β	Constants defined in Equation (2.5).
γ_n	Constants defined in Equation (2.3).
ε	Measure of nonlinearity in dissipation function.
ζ	Linear damping ratio ($\zeta = C_1/2m\omega_n$).
Θ_o	Phase angle.
$\Theta(t)$	Time varying phase angle.
λ	Stepping parameter in search routine.
λ_s	Initial value of stepping parameter.
τ	Golden Section reduction factor ($\tau = 0.618$).
$\phi_n(t)$	Successive terms in Perturbation Series Approximation for $n = 0,1,2,\cdots$.
$\psi(t)$	Correction term from Extended K-B Approximation.
ω_n	Undamped natural frequency ($\omega_n = \sqrt{K/m}$).
ω_d	Damped natural frequency ($\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$).

SUMMARY

An investigation of the dynamic response of a nonlinear system has been conducted. The particular case of a system, whose nonlinear terms appear in its dissipation function, has been considered. The nonlinear damping function was represented as a general polynomial in the system's velocity. Therefore, the dissipation function consisted of the combination of a Coulomb friction term, a linear, velocity proportional term, and nonlinear terms proportional to higher powers of the velocity.

The research dealt with the specific case of a single degree of freedom system and consisted of two areas of investigation. Two forms for the approximate representation of the solution of the system's equation of motion were formulated. The first of these was an extension of the classic Kryloff-Bogoliuboff approximation technique. The second was a further application of a perturbation series approximation. No limitations were imposed on the allowable magnitudes of either the Coulomb or viscous damping terms in these approximations. Hence, subsident or oscillatory system response may be represented with these approximation techniques.

In the second phase of the investigation, a method was developed for converting the measured dynamic response of a system into estimates of its unknown dissipation function. A computational algorithm employing an optimization technique was written for determining the unknown system parameters. This procedure used a least squares approach to determine the measure of agreement between the mathematical representation of the system and the physical system. The response of the mathematical system may be expressed either in terms of an approximation representation or through a numerical integration technique. Application of this technique was demonstrated with data obtained for a physical system and also mathematically generated response information.

CHAPTER I

INTRODUCTION

Object of the Investigation

It is often the case when investigating a dynamic system, that linear theory is not adequate to describe the response of the system. Quite often a physical system exists and an acceptable mathematical model also seems to exist, but the system's parameters are unknown. This investigation develops methods for converting observations on the system into estimates of its unknown parameters. Also, techniques have been developed for constructing an approximate representation of the system's response once these parameters are known. These procedures have been applied to observations obtained from a physical system and also to mathematically generated response information.

The particular problem which has been considered in this study is that of a single degree of freedom system which possesses linear restoring forces and a nonlinear dissipation function. The nonlinear damping force was represented as a general polynomial in velocity. Hence, the dissipation function consisted of the combination of a Coulomb friction term, a linear, velocity proportional term, and nonlinear terms proportional to higher powers of the velocity. The governing differential equation of motion for such a system (see Figure 1) is as follows:

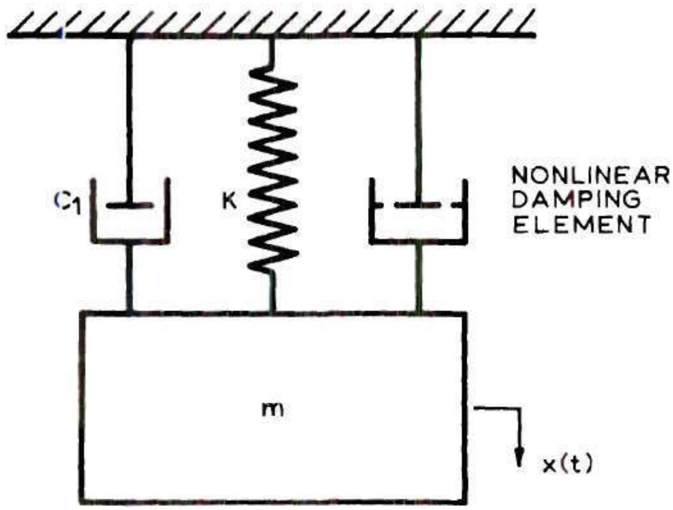


Figure 1. Single Degree of Freedom System

$$m\ddot{x} + C_o Sgn(\dot{x}) + C_1\dot{x} + F(\dot{x}) + Kx = 0 \qquad (1.1)$$

where

x = displacement.

$\dot{x}$ = velocity.

$\ddot{x}$ = acceleration.

m = mass.

K = spring rate.

C_n = damping coefficients for $n = 0,1,2,\ldots$.

$F(\dot{x}) = C_2 Sgn(\dot{x})\dot{x}^2 + C_3\dot{x}^3 + \cdots$

$$Sgn(\dot{x}) = \begin{cases} +1 \text{ for } \dot{x} > 0 \\ -1 \text{ for } \dot{x} < 0 \end{cases} .$$

Note that the total dissipation function is odd in nature, that it always absorbs energy from the system, and that it is zero when the velocity is zero.

Investigation Procedure

Two methods have been developed by which an approximate expression for the response of a system governed by Equation (1.1) may be constructed. Of course, these approximate methods placed some restrictions on the permissible magnitudes of the nonlinear damping coefficients which could be included in the system. The first of these techniques was an extension of the classic Kryloff-Bogoliuboff approach. Here, the assumed form of the system response was such that it more closely reflected the magnitudes of the Coulomb and viscous damping terms. This analysis was more complete than the previous Kryloff-Bogoliuboff methods because no restrictions were placed on these two coefficients. A second approximation, comprising a power series expansion in terms of the solution of the linear differential equation, was also formulated. As before, no limitations on the viscous or Coulomb damping coefficients were in effect.

Experimental data were obtained from a damped, single degree of freedom, spring-mass model. With this model it was possible to consider systems with various combinations of damping, spring rate, mass, and initial conditions. In all of the experimental work, the displacement of the mass was determined with a light sensitive displacement transducer.

A computational optimization method, known as an Optimum Search Procedure, was developed for determining the unknown system parameters. Here, the response of a mathematical model with an assumed damping law was compared with the response from the physical system. Techniques were developed by which the damping law could be altered in order to obtain the "best fit" between the two responses. The measure of this "fit," or criterion function as it was called, employed a classical least squares comparison between the measured and analytical responses.

These methods which have been developed for determining the unknown damping parameters differ from the commonly used approach to the problem. Quite often the force dissipated is measured in terms of the velocity and then the unknown constants are determined. By using the system's transient displacement-time history, the experimental work required was minimized, but the analytical considerations were more complex in nature. However, one purpose of the research was to develop methods which took advantage of modern, high-speed computers in conjunction with easily obtained physical data.

Historical Background

This investigation is an outgrowth of the work that has been conducted by Baumgarten and Kitchen (1) and Baumgarten, Wirth, and Watson (2). They were interested in the analysis of a hydraulic shock absorber in conjunction with the design of a printer for a digital computer. As indicated in the above references, the description of this shock absorber as a linear subsident or aperiodic system did not adequately simulate the physically measured response. These works pointed out the need for further investigations of a subsident system with nonlinear damping forces present.

The application of various segments of Equation (1.1) form the basis of large areas of engineering studies. The analysis and solution for these forms are well documented in the literature. For instance, the large field of single degree of freedom, autonomous, linear vibration theory is governed by the following differential equation:

$$m\ddot{x} + C_1\dot{x} + Kx = 0$$

There is a vast wealth of literature dealing with the many aspects of the application of this equation.

Another fairly common form of Equation (1.1) is that in which Coulomb friction or Coulomb plus linear damping are present. This case is characterized by the following equation of motion:

$$m\ddot{x} + C_o\mathrm{Sgn}(\dot{x}) + C_1\dot{x} + Kx = 0$$

In both of these cases, the problem is nonlinear over the entire range of time. However, the equation is linear inside certain time intervals. The problem can be solved within these time intervals and then the total solution obtained by fitting together the results over each of these segments.

These last two forms of Equation (1.1) have been studied by several investigators, a few of which are given here. Cunningham (3), Timoshenko and Young (4), Hansen and Chenea (5), Jacobsen and Ayre (6) and Pipes (7) have considered the case of Coulomb friction alone. They have obtained closed form expressions for the transient response of this system. These analyses have been extended to include both Coulomb and viscous damping by Jacobsen and Ayre (6), Pipes (7), Bögel (8), and Rubbert (9).

Bogusz (10), Lewandowski (11), Ziemba (12) and Milne (13) have developed a number of theorems and techniques for the qualitative analysis of systems with nonlinear dissipation functions. They have discussed methods of predicting the types of motion which would result with various system constants and initial conditions. Bogusz has developed a method of determining the requirements for subsident motion. Lewandowski's paper treated a system with a combination of nonlinear damping and restoring force terms while Ziemba was concerned with only nonlinear damping terms. They both discussed the phase plane representation of these various systems. Milne considered a system whose dissipation function was dependent on various combinations of its velocity. The type of motion which results depends on the magnitude of the linear damping term. He shows that if this term is greater than critical, the motion will be subsident in nature. However, as is shown in Appendix A, this is not the case for a system with Coulomb friction present. For such a system, subsident motion can result even though the linear damping is less than critical, provided that the Coulomb term is sufficiently large.

Considerable work has also been done with a single degree of freedom system which possesses a single damping term proportional to the square of the velocity. The governing equation for such a system is

$$m\ddot{x} + C_2 \mathrm{Sgn}(\dot{x})\dot{x}^2 + Kx = 0 \tag{1.2}$$

Bogusz and Kazimierz (14) and Butenin (15) have presented procedures for the construction of phase plane plots for this system. They did not deal with the problem in great detail, but rather have outlined the graphical construction for this specific example. Stoker (16), Ku (17), and Struble (18) have also considered the phase plane construction for a system similar to that of Equation (1.2). They considered the case of a velocity squared damping force acting on a pendulum. Of course, in this example the restoring force is also nonlinear in nature.

It is possible to obtain an exact expression for the first integral of Equation (1.2) with only velocity squared damping present. From this, a relationship between the system's velocity and its displacement may be obtained. Mises (19), Klotter (20,21), and Magnus (22) have employed this approach to determine the ratio of the successive amplitudes. These relationships were displayed as graphs which gave the amplitudes of motion at the points of zero velocity. Several investigators have used various approximate techniques after obtaining this first integral. Ignatowsky (23), Grammel (24), and Pöschl (25) expanded the resulting exponential function as a truncated power series. In this manner, they obtained an expression for the ratio of the successive amplitudes of motion. By expanding the argument of the exponential function as a series in ascending powers of the amplitude, Richardson (26) was able to obtain a solution to Equation (1.2). Beginning with the relation between the displacement and velocity, Van Zandt (27) set up the general integral equation relating the displacement to the time. He was not able to integrate this equation, but after expanding it according to the binomial theorem, he developed an approximate expression for the response.

Various miscellaneous approximation techniques have been discussed for handling dynamic systems, such as Equation (1.1). Klotter (20), p. 171, and Morley and Bryce (28) have employed an energy approach to the problem. They assumed that the frictional work done in any one cycle must be balanced by the loss in strain energy. From this energy balance, an equivalent viscous damping term was obtained. Richardson (29) and Plato (30) assumed relationships between the successive amplitudes of decay which yielded expressions for the time varying amplitude of motion. Routh (31) applied a technique which he referred to as a "continued approximation." He first derived the solution to the linearized equation of motion. These results were substituted into the nonlinear terms and were treated as time dependent disturbing forces. The solution to this equation was determined and was treated as a corrected representation of the response. Milne (13,32) presents a number of tables which furnish values for the successive amplitudes. These ratios were obtained by means of a series expansion and the case of velocity squared and velocity squared plus viscous damping have been presented.

An approximate method commonly known as the Kryloff-Bogoliuboff Method (33) has been applied to various forms of Equation (1.1) by several investigators. Reference 33 is the basic reference for this technique and references 34 through 37 present general discussions concerning the application of this approximation to systems with a nonlinear dissipation function. The approach assumes that the solution to the differential equation is of the form

$$x(t) = X(t)\text{Sin}(\omega t + \Theta(t))$$

and

$$\dot{x}(t) = -\omega X(t)\text{Cos}(\omega t + \Theta(t))$$

where the amplitude and phase angle are not constants as is the case in the linear problem. Rather, these terms are assumed to be functions of time to account for the nonlinearities present in the system. The specific configuration governed by Equation (1.2) has been discussed by Minorsky (34) and also in references 36 through 41. In addition, Minorsky (34), Klotter (41), and Brunelle (42) have considered the case containing the combination of a linear and a velocity squared damping term. Minorsky and Klotter's results require that both the linear and the nonlinear damping terms be very small. By including the effect of the viscous damping coefficient in a damped natural frequency expression, Brunelle removed some of the restriction on the magnitude of this linear coefficient.

A second approximation technique which has found common application in nonlinear problems is known as the perturbation series approach. Here, the response of the system is expressed as a power series. This technique may be applied in cases where there is a parameter of small magnitude associated with the nonlinear terms. Cunningham (3), pp. 123-133, Minorsky (37), pp. 217-231, Bellman (43, 44), and Nayfeh (45) present general discussions on this approximation method. Pipes (46) has applied this technique to a system with velocity squared damping. He performed the analysis in conjunction with the use of Laplace transforms.

Very little work has been done in the area of obtaining either approximate solutions for Equation (1.1) or the evaluation of the nonlinear damping constants from experimental records when a system is governed by such an equation. Klotter (41) did consider the case of a combination of viscous and velocity squared damping terms, but only when the system was very lightly damped. Eggleston and Mathews (47) have assembled a good review of techniques which might be applied to a linear system.

The field of controls engineering has been concerned with the identification of system parameters. Often, their approach involves determining the characteristic variational problem and then developing an iterative technique for the solution of the resulting Euler-Lagrange differential equations. These approaches to the question require the linearization of the original problem at some point in the analysis.

Several papers (48,49,50) have used a technique known as quasilinearization to obtain linear differential equations which describe the unknown parameters in terms of observations made on the system response. These equations were then solved in a sequential manner to obtain the unknown system constants. Detchmendy and Sridar (51) have proposed a method of sequential estimation applicable to continued observation of plant response.

In the above references, the methods developed apply to a system with a known, fixed form describing function. The problem which has been discussed is one of identifying some of the system parameters and state variables. In the current investigation, the exact form of the equation of motion was unknown. In addition, with the approach presented here, the predicted response has been matched with the observed data over a time interval beginning at time zero. This is of importance because the form of the response of a nonlinear system is dependent on the magnitudes of the initial conditions. This approach was not the case in the above papers.

In addition, Hsieh (52) and Balakrishnan (53) have characterized the dynamic system through the concept of a functional. Here, the system's output was expressed as a functional of the input. This expression was then expanded as a functional power series and procedures for evaluating the parameters developed. Diamessis (54,55,56) has approached the question in an entirely different manner. He reduces the original problem to that of solving a set of linear algebraic equations for the unknown system parameters. This approach requires the integration of the measured response a number of times equal to the order of the governing differential equation. However, this did not seem to cause problems since integration is a smoothing operation.

A variety of computational techniques known as optimum search procedures have been developed in the past few years. These methods have been used by applied mathematicians to obtain the solution of a group of simultaneous equations. Also, the field of chemical engineering has applied these techniques to the evaluation and design of various chemical processes. All of these procedures define methods for determining a set of independent variables which yield a minimum (or maximum) of a function which has been called among other things, the cost function, payoff function, object function, or criterion function.

Examples of early work in this area are given by Box and Wilson (57) and Booth (58). This first paper is quite extensive and develops some of the basic ideas behind the so-called method of steepest descent. References 59, 60, 61, and 62 present discussions and reviews on the application of various descent or gradient optimization procedures. Dixon (63) gives a very concise outline of the true method of steepest descent. The work by Wilde (64), Carnahan and Wilkes (65), and Lasdon and Warren (66) was very helpful in the development of the optimization techniques employed in conjunction with this investigation.

It should be noted that in the above discussions the criterion function could be expressed in some continuous, closed form manner with respect to the unknown parameters. This allowed differentiation and other mathematical operations to be performed on the criterion function. Since the equation of motion of the system considered here was non-linear, a closed form criterion function could not be obtained. Also, the measured response was not expressible as a continuous function, but rather was available only at discrete points in time.

CHAPTER II

ANALYTICAL INVESTIGATION

Introduction

Various techniques for generating an approximate solution to Equation (1.1) have been considered. These approximations placed no restriction on the allowable magnitudes of the Coulomb and viscous damping coefficients, but the permissible range of the higher order damping terms was limited. The forms of the various approximations were grouped according to the amount of linear damping in the system. The two cases of viscous damping that were considered are: (1) viscous damping greater than critical and (2) viscous damping less than critical.

These approximations are a definite extension of the work found in the literature. No approximate solutions for a nonlinear system with subsident type motion was found. In addition, some of the limitations placed on the oscillatory case were removed by not having to restrict the magnitudes of the Coulomb and viscous coefficients.

For reference purposes, Equation (1.1) is repeated here.

$$m\ddot{x} + C_o \mathrm{Sgn}(\dot{x}) + C_1\dot{x} + F(\dot{x}) + Kx = 0 \qquad (1.1)$$

where

$$F(\dot{x}) = C_2 \mathrm{Sgn}(\dot{x})\dot{x}^2 + C_3\dot{x}^3 + \cdots$$

This equation is known as a quasi-linear differential equation when the nonlinear term $F(\dot{x})$ is "small." When $F(\dot{x})$ and C_o are both zero, Equation (1.1) reduces to the well-known linear equation

$$m\ddot{x} + C_1\dot{x} + Kx = 0$$

In all of the approximate solutions which were obtained, the following initial conditions were assumed:

$$x(0) = X_o \qquad \text{and} \qquad \dot{x}(0) = 0 \tag{2.1}$$

In each of the viscous damping cases, two forms for the approximate solution were obtained. The first, referred to as the Extended K-B Approximation, was a modification of the commonly known Kryloff-Bogoliuboff technique. The second approximate solution, referred to as the Perturbation Series Approximation, employed a power series approximation for the solution of Equation (1.1).

The results of the various approximation techniques were compared with the response obtained by numerically integrating the governing equation of motion. These comparisons were carried out for a system with the appropriate damping coefficients, various initial conditions, and the following mass and spring rate:

$$m = 1.0 \text{ lb-sec}^2/\text{in}$$

$$K = 100 \text{ lb/in}$$

It was assumed that the numerical solution, x_n, was correct and the approximate solution, x_a, was compared to it by

$$\text{Per Cent Error} = \frac{x_a - x_n}{x_n} \times 100 \tag{2.2}$$

Equation (1.1) is rewritten as follows in order to simplify the notation in the subsequent discussions:

$$\ddot{x} + \frac{C_o}{m} \mathrm{Sgn}(\dot{x}) + 2\zeta\omega_n \dot{x} + \varepsilon f(\dot{x}) + \omega_n^2 x = 0 \tag{2.3}$$

where

$$\varepsilon = \mathrm{Max}\left(\frac{C_i}{m}\right) \quad \text{and} \quad \gamma_i = \frac{1}{\varepsilon}\left(\frac{C_i}{m}\right) \quad \text{for } i = 2,3,4,\cdots$$

and

$$f(\dot{x}) = \gamma_2 \mathrm{Sgn}(\dot{x})\dot{x}^2 + \gamma_3 \dot{x}^3 + \cdots$$

In this equation, ε is the measure of the magnitude of the nonlinear damping function $f(\dot{x})$.

Extended K-B Approximation

Approximation Technique

The form of the solution for the simple linear equation of motion is

$$x(t) = X\mathrm{Sin}(\omega t + \Theta)$$

and

$$\dot{x}(t) = -\omega X \mathrm{Cos}(\omega t+\Theta)$$

where X and Θ are constants and are the amplitude and phase angle, respectively. The classic Kryloff-Bogoliuboff approximation (33,34) retains the form of the above solution, but considers the quantities X and Θ as unknown functions of time which are to be determined. Therefore, the Kryloff-Bogoliuboff technique assumes that the solution to the nonlinear equation of motion has the form

$$x(t) = X(t)\mathrm{Sin}(\omega t+\Theta(t))$$

and

$$\dot{x}(t) = -\omega X(t)\mathrm{Cos}(\omega t+\Theta(t))$$

A technique has been developed which allows this type of approximation to be extended to a system with a more general dissipation function than before.

In the Extended K-B Approximation, the form of the assumed solution was derived from the form the solution would assume if the equation of motion were linear. First order differential equations for the introduced time varying coefficients were then obtained. The Fourier series expansions of the usual Kryloff-Bogoliuboff analysis were omitted. Instead, assumptions as to the form of these coefficient equations were made which yielded closed form expressions for the quantities.

Due to the nature of the assumptions that were made, the magnitudes of the viscous and Coulomb damping terms were not limited in this approximation. The case of a highly damped, subsident system can be handled, in addition to the more familiar, lightly damped, oscillatory problem. Also, an underdamped system with a relatively high damping ratio of 0.8 or 0.9, can be analyzed.

Overcritical Viscous Damping

Mathematical Derivation. When the linear damping term is greater than the critical value, the resulting motion will be subsident or aperiodic in nature. The displacement and velocity for such a highly damped system when $f(\dot{x})$ is zero and with the initial conditions given in Equation (2.1) are

$$x(t) = \frac{X^*}{\alpha - \beta}\left(\alpha e^{\beta t} - \beta e^{\alpha t}\right) - \frac{C_o}{m\omega_n^2}\,\mathrm{Sgn}(\dot{x}) \qquad (2.4)$$

and

$$\dot{x}(t) = \frac{\alpha\beta}{\alpha - \beta} X^*\left(e^{\beta t} - e^{\alpha t}\right)$$

Here, X^* is the amplitude of the motion corrected for the presence of Coulomb friction. In these expressions, α and β are defined as

$$\alpha = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n \quad \text{and} \quad \beta = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n \qquad (2.5)$$

It was now assumed that the displacement expression could be modified as follows

$$x(t) = \frac{X(t)}{\alpha - \beta}\left(\alpha e^{\beta t} - \beta e^{\alpha t}\right) - \frac{C_o}{m\omega_n^{\ 2}}\, \mathrm{Sgn}(\dot{x}) \tag{2.6a}$$

Consistent with the Kryloff-Bogoliuboff approximation, the form of the modified velocity expression was forced to match that of the linear case. Therefore we have for the velocity

$$\dot{x}(t) = \frac{\alpha\beta}{\alpha - \beta}\, X(t)(e^{\beta t} - e^{\alpha t}) \tag{2.6b}$$

From the above, the acceleration becomes

$$\ddot{x}(t) = \frac{\alpha\beta}{\alpha - \beta}\, [X(t)(\beta e^{\beta t} - \alpha e^{\alpha t}) + \dot{X}(t)(e^{\beta t} - e^{\alpha t})]$$

Inserting the above expression, together with Equations (2.6), into Equation (2.3) and collecting terms yields

$$\frac{dX}{dt} = -\frac{\varepsilon(\alpha-\beta)}{\alpha\beta}\ \frac{f(\dot{x})}{(e^{\beta t} - e^{\alpha t})} \tag{2.7}$$

The solution to this differential equation results in the amplitude of the motion which is modified by the presence of the nonlinear damping terms.

Example of Application. Equation (2.7) can be solved when only one nonlinear damping term is present in the dissipation function.

Otherwise, the resulting differential equation is nonlinear and a closed form solution is not obtainable. The following is an example of the application of Equation (2.7) to a system which possesses a velocity squared term in addition to Coulomb and viscous damping.

Since the response will be subsident in nature, the velocity will always have the same sign and the dissipation function may be written as

$$f(\dot{x}) = \frac{1}{\varepsilon} \frac{C_2}{m} \dot{x}|\dot{x}|$$

Using the assumed form of the velocity and substituting the above into Equation (2.7) results in

$$-\int_{X(o)}^{X(t)} \frac{dX}{X^2} = \int_{o}^{t} \varepsilon\left|\frac{\alpha\beta}{\alpha - \beta} (e^{\beta t} - e^{\alpha t})\right| dt$$

When this relationship is integrated and combined with Equation (2.6a), the response is given by

$$x(t) = \frac{X(o)(\alpha e^{\beta t} - \beta e^{\alpha t})}{(\alpha-\beta)\{1+\varepsilon X(o)[\psi(t)-\psi(o)]\}} + \frac{C_o}{m\omega_n^2} \tag{2.8}$$

where

$$\psi(t) = \frac{1}{\alpha - \beta}\left(\alpha e^{\beta t} - \beta e^{\alpha t}\right)$$

$$\varepsilon = \frac{C_2}{m}$$

and

$$X(o) = X_o - \frac{C_o}{m\omega_n^2}$$

The above result is consistent with the initial conditions given in Equation (2.1).

Figures 2, 3, and 4 show the results of this approximation technique with a velocity squared term present. Figure 2 indicates the percentage error as a function of time for various values of ε. These results are for a linear damping ratio of 2.0. The error in this approximation also depends on the magnitudes of both the linear damping coefficient and the initial displacement. The percentage error becomes much greater as the magnitude of the initial displacement increases. This trend is indicated in Figure 3. In Figure 4, it is seen that the error is fairly constant in terms of the viscous damping. These last two curves are drawn for a time at which point the system's displacement is two-thirds of its initial displacement.

Undercritical Viscous Damping

Mathematical Derivation. A dynamic system which possesses a small amount of linear damping will be characterized by an oscillatory type motion. The displacement and velocity for such a lightly damped linear system with the assumed initial conditions are

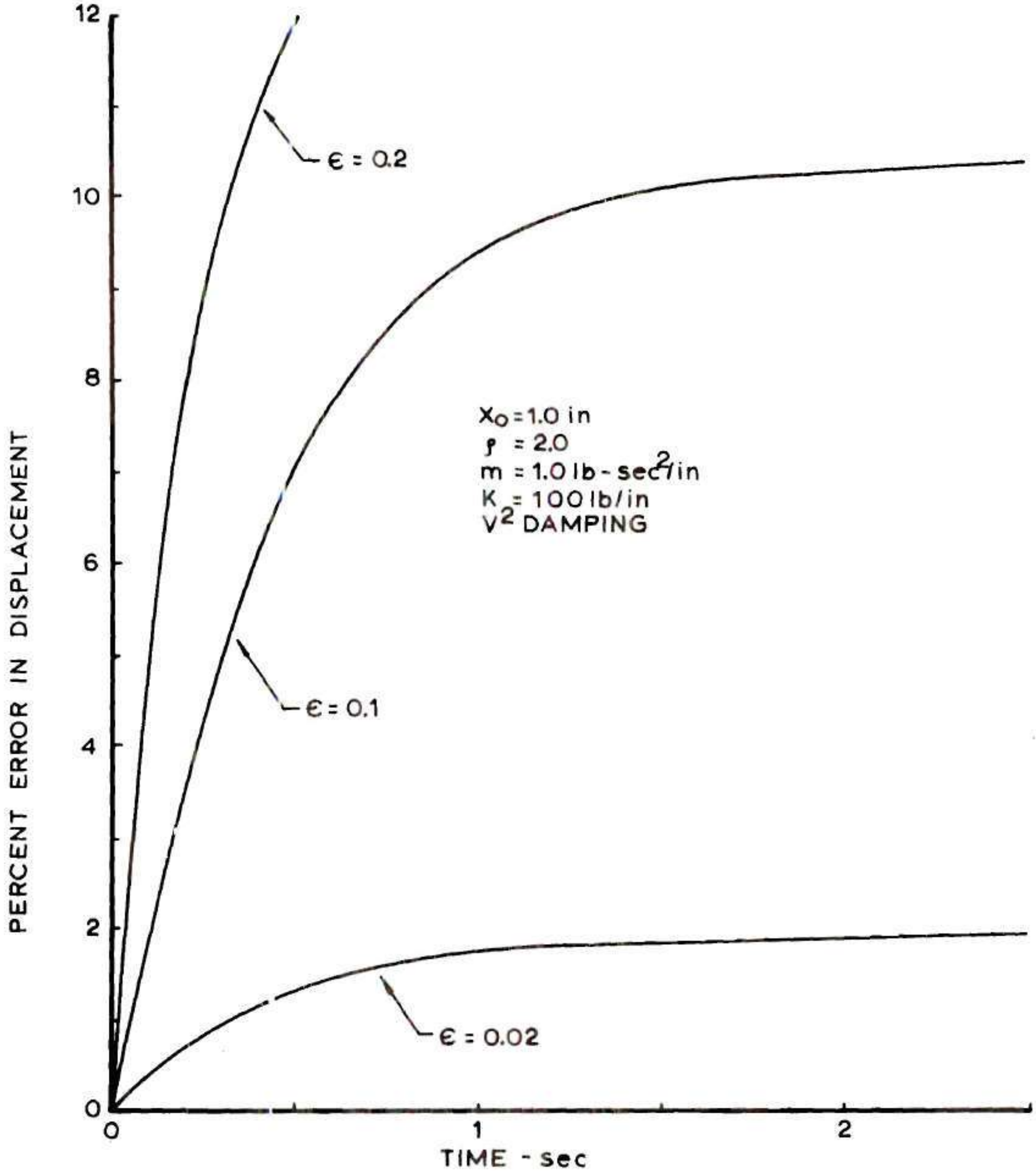


Figure 2. Error in Displacement vs. Time—Extended K-B Approximation—$\zeta > 1$

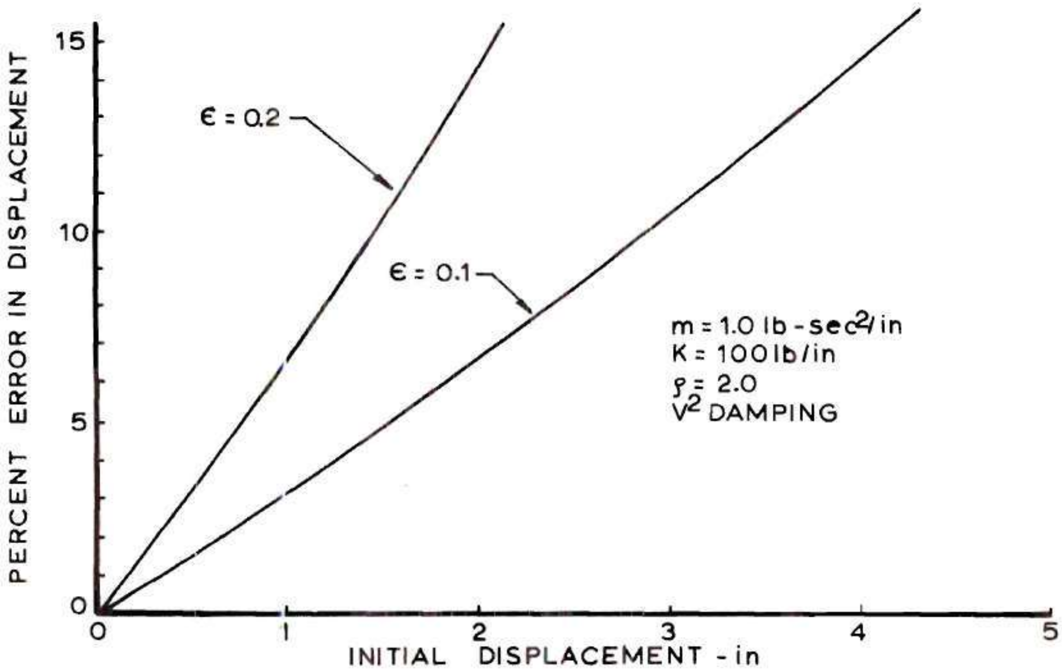


Figure 3. Error in Displacement vs. Initial Displacement—Extended K-B Approximation—$\zeta > 1$

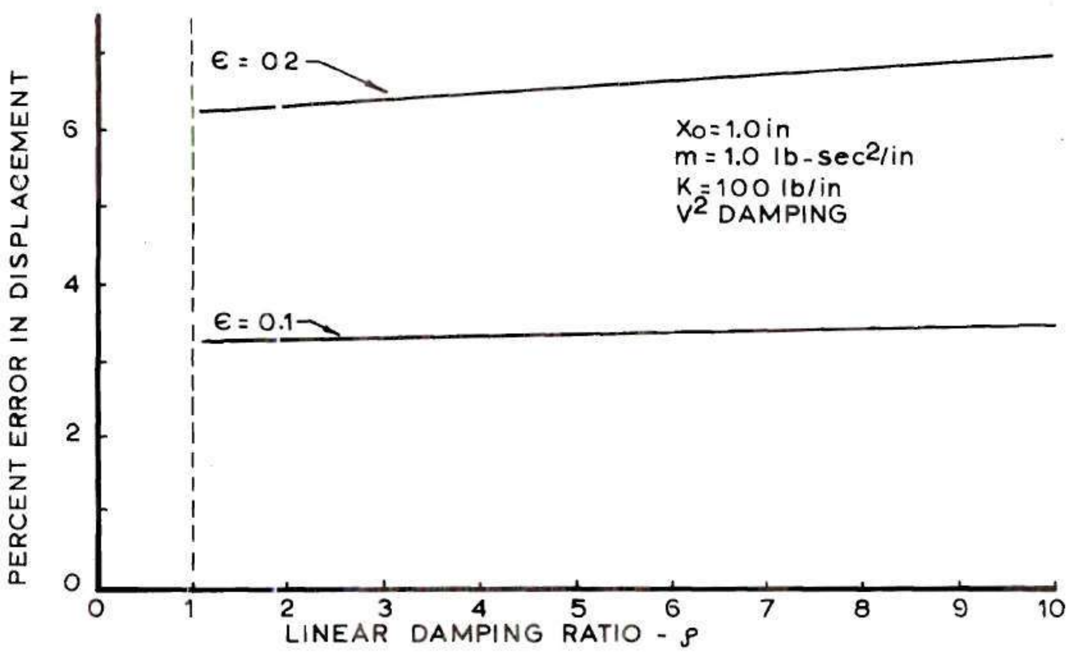


Figure 4. Error in Displacement vs. Linear Damping Ratio—Extended K-B Approximation—$\zeta > 1$

$$x(t) = X^{*} e^{-\zeta\omega_n t} \text{Sin}(\omega_d t+\Theta) - \frac{C_o}{m\omega_n^2} \text{Sgn}(\dot{x}) \tag{2.9}$$

$$\dot{x}(t) = X^{*} e^{-\zeta\omega_n t} [-\zeta\omega_n \text{Sin}(\omega_d t+\Theta) + \omega_d \text{Cos}(\omega_d t+\Theta)]$$

where ω_d is the damped natural frequency.

A time varying amplitude, $X(t)$, and phase angle, $\Theta(t)$, were introduced into Equation (2.9) in order to account for the nonlinear terms in the dissipation function. The modified displacement is expressed as

$$x(t) = X(t) e^{-\zeta\omega_n t} \text{Sin}(\omega_d t+\Theta(t)) - \frac{C_o}{m\omega_n^2} \text{Sgn}(\dot{x}) \tag{2.10}$$

When the above equation is differentiated with respect to time, the following is obtained

$$\dot{x}(t) = \dot{X}(t) e^{-\zeta\omega_n t} \text{Sin}(\omega_d t+\Theta(t)) + X(t) e^{-\zeta\omega_n t} [-\zeta\omega_n \text{Sin}(\omega_d t+\Theta(t))$$

$$+ (\omega_d+\dot{\Theta}(t)) \text{Cos}(\omega_d t+\Theta(t))]$$

As is done in the Kryloff-Bogoliuboff technique, it was assumed that the form of the velocity expression was to be the same as that of the linear case. Therefore, the following condition was imposed on the velocity expression.

$$\dot{X}(t)e^{-\zeta\omega_n t}\text{Sin}(\omega_d t+\Theta(t)) + X(t)\dot{\Theta}(t)e^{-\zeta\omega_n t}\text{Cos}(\omega_d t+\Theta(t)) = 0 \quad (2.11)$$

With the restriction of Equation (2.11) in effect, the velocity and acceleration of the system become

$$\dot{x}(t) = X(t)e^{-\zeta\omega_n t}[-\zeta\omega_n\text{Sin}(\omega_d t+\Theta(t)) + \omega_d\text{Cos}(\omega_d t+\Theta(t))]$$

$$\ddot{x}(t) = (\dot{X}(t)-\zeta\omega_n X(t))e^{-\zeta\omega_n t}[-\zeta\omega_n\text{Sin}(\omega_d t+\Theta(t)) +$$

$$+ \omega_d\text{Cos}(\omega_d t+\Theta(t))] - X(t)e^{-\zeta\omega_n t}(\omega_d+\dot{\Theta}(t))[\zeta\omega_n\text{Cos}(\omega_d t+\Theta(t)) +$$

$$+ \omega_d\text{Sin}(\omega_d t+\Theta(t))]$$

Combining the above expressions for the displacement, velocity, and acceleration with the equation of motion, Equation (2.3), and incorporating the condition imposed by Equation (2.11), the following is obtained

$$\dot{\Theta}(t) = \frac{\varepsilon}{\omega_d X(t)}e^{\zeta\omega_n t}f(\dot{x})\text{Sin}(\omega_d t+\Theta(t)) \quad (2.12a)$$

and

$$\dot{X}(t) = -\frac{\varepsilon}{\omega_d}e^{\zeta\omega_n t}f(\dot{x})\text{Cos}(\omega_d t+\Theta(t)) \quad (2.12b)$$

The solution of these differential equations gives the effect of the nonlinear damping terms present. In this case, two equations result, describing the time rate of change of both the amplitude and the phase angle.

Example of Application. Again, the effect of a velocity squared damping term was investigated. With a lightly damped linear oscillator, the damping forces have little effect on the frequency of response. At this point, it was assumed that the same type of phenomenon exists with the lightly damped nonlinear system. Thus, only the viscous term, which might be relatively large, was allowed to contribute to the modification of the system's period. Therefore, Equations (2.12) reduce to

$$\dot{\Theta}(t) = 0$$

$$\dot{X}(t) = -\frac{\varepsilon}{\omega_d} e^{\zeta\omega_n t} f(\dot{x})\mathrm{Cos}(\omega_d t+\Theta_o)$$

The dissipation function in this example is

$$f(\dot{x}) = \frac{1}{\varepsilon}\frac{C_2}{m}\,\mathrm{Sgn}(\dot{x})\dot{x}^2$$

Substituting this expression into the above differential equation for the amplitude and collecting terms gives

$$-\int_{X(o)}^{X(t)} \frac{dX}{X^2} = \int_o^t \frac{\varepsilon}{\omega_d}\,\mathrm{Sgn}(\dot{x})e^{-\zeta\omega_n t}[-\zeta\omega_n \mathrm{Sin}(\omega_d t+\Theta_o) + \omega_d \mathrm{Cos}(\omega_d t+\Theta_o)]^2 \mathrm{Cos}(\omega_d t+\Theta_o)dt$$

Equation (2.13) describes the response of the mass when integrating the above over the first half period. The fact that the function $Sgn(\dot{x})$ changes sign, requires that the total response be obtained by fitting together the solutions over consecutive half periods.

$$x(t) = X(t)e^{-\zeta\omega_n t}\,Sin(\omega_d t+\Theta_o) - \frac{C_o}{m\omega_n^2}\,Sgn(\dot{x}) \qquad (2.13)$$

where

$$X(t) = \frac{X(o)}{1 + \varepsilon X(o)[\psi(t)-\psi(o)]}$$

$$\psi(t) = \frac{Sgn(\dot{x})e^{-\zeta\omega_n t}}{3\omega_d(9-8\zeta^2)}\,[\zeta\omega_n(15-11\zeta^2)Cos^3(\omega_d t+\Theta_o) + 9\zeta^2\omega_d Sin^3(\omega_d t+\Theta_o)$$

$$+ 3(3-5\zeta^2)\omega_d Sin(\omega_d t+\Theta_o)Cos^2(\omega_d t+\Theta_o) - 3\zeta^3\omega_n Sin^2(\omega_d t+\Theta_o)Cos(\omega_d t+\Theta_o)$$

$$+ 6\zeta\omega_n(4\zeta^2-3)(1-\zeta^2)Cos(\omega_d t+\Theta_o) + 6\omega_d(2\zeta^2-3)(2\zeta^2-1)Sin(\omega_d t+\Theta_o)]$$

$$X(o) = \frac{X_o + \frac{C_o}{m\omega_n^2}\,Sgn(\dot{x})}{Sin\Theta_o}$$

$$\Theta_o = Tan^{-1}\,\frac{\sqrt{1-\zeta^2}}{\zeta}$$

and

$$\varepsilon = \frac{C_2}{m}$$

The results of this form of the approximation are given in Figures 5 through 10. All of these curves are for a system with a nonlinear velocity squared term present in the dissipation function. The percentage error in the displacement is shown for the amplitudes at the end of the first and second damped periods. In this case, the damped period is defined as that portion of the response between two successive positive extremes. In addition, to indicate the validity of the assumption made that the time response is not affected by the nonlinear term, the error associated with the damped period of the system is shown in Figures 6 and 10. These graphs indicate the error between the time, as predicted by the approximation, when the system's response goes through zero and that obtained from the numerical solution. This error is expressed as a percentage in terms of the time obtained from the numerical response information. In these curves, the term "1st Zero" refers to the time when the system's response first goes through zero after being released. The time represented by the term "2nd Zero," is the location of the following point of zero displacement.

Perturbation Series Approximation

Approximation Technique

A second approximate representation of the system's motion was obtained through the application of the perturbation series technique. This method is applicable when a small parameter is associated with the nonlinear damping terms. The approximate solution was written as a power series in the displacement, with the terms of the series involving

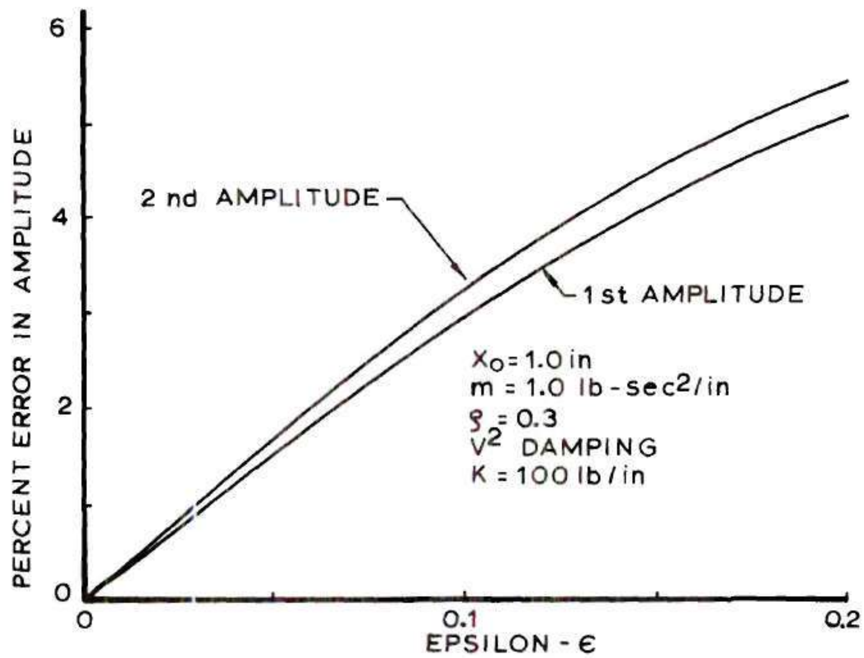


Figure 5. Error in Amplitude vs. Epsilon--Extended K-B Approximation--$\zeta < 1$

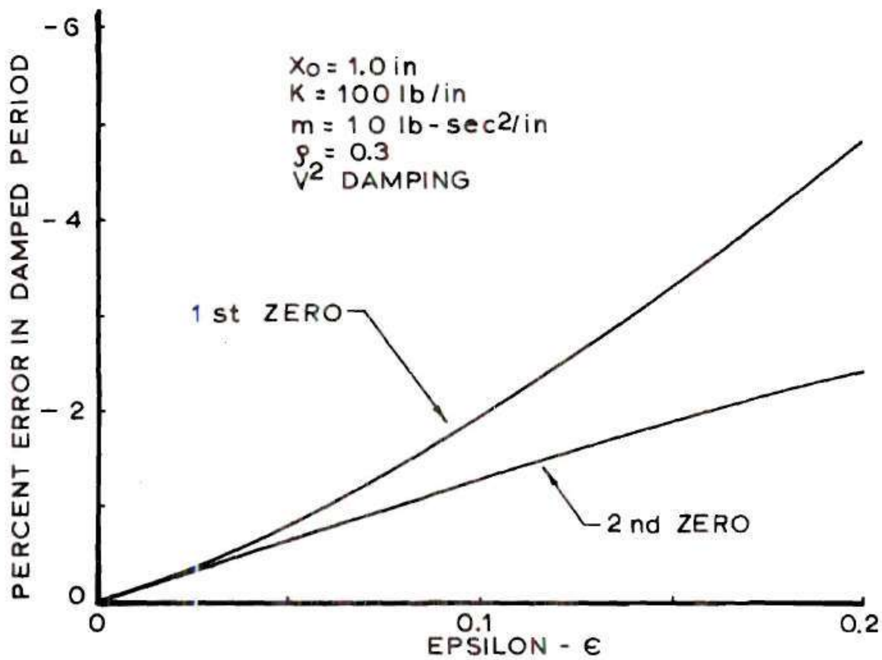


Figure 6. Error in Damped Period vs. Epsilon--Extended K-B Approximation--$\zeta < 1$

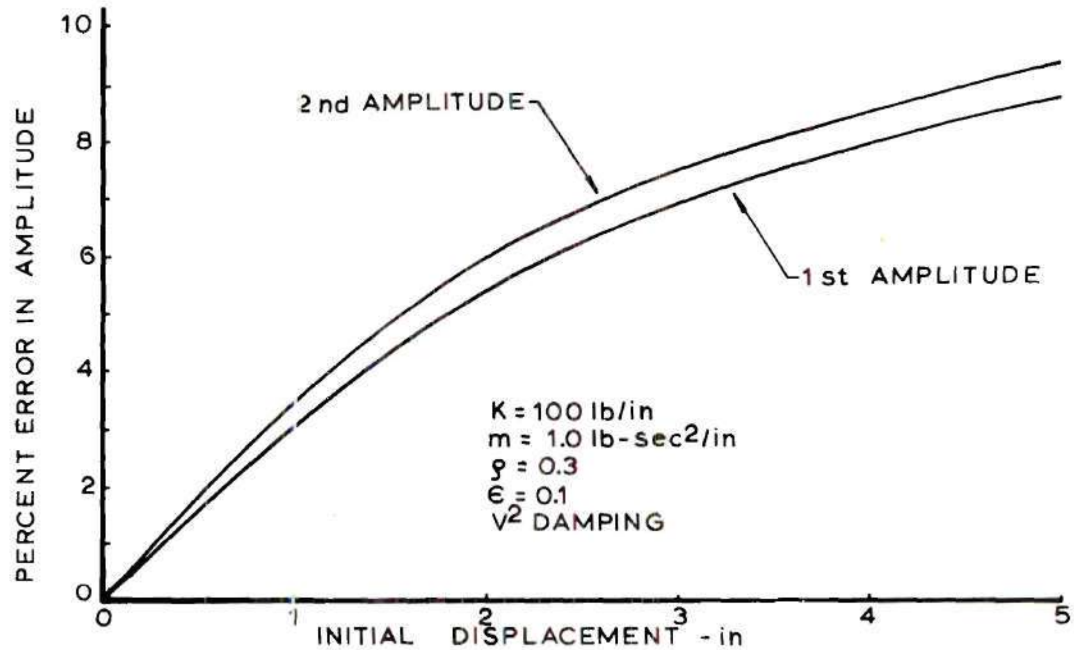


Figure 7. Error in Amplitude vs. Initial Displacement--Extended K-B Approximation--$\zeta < 1$

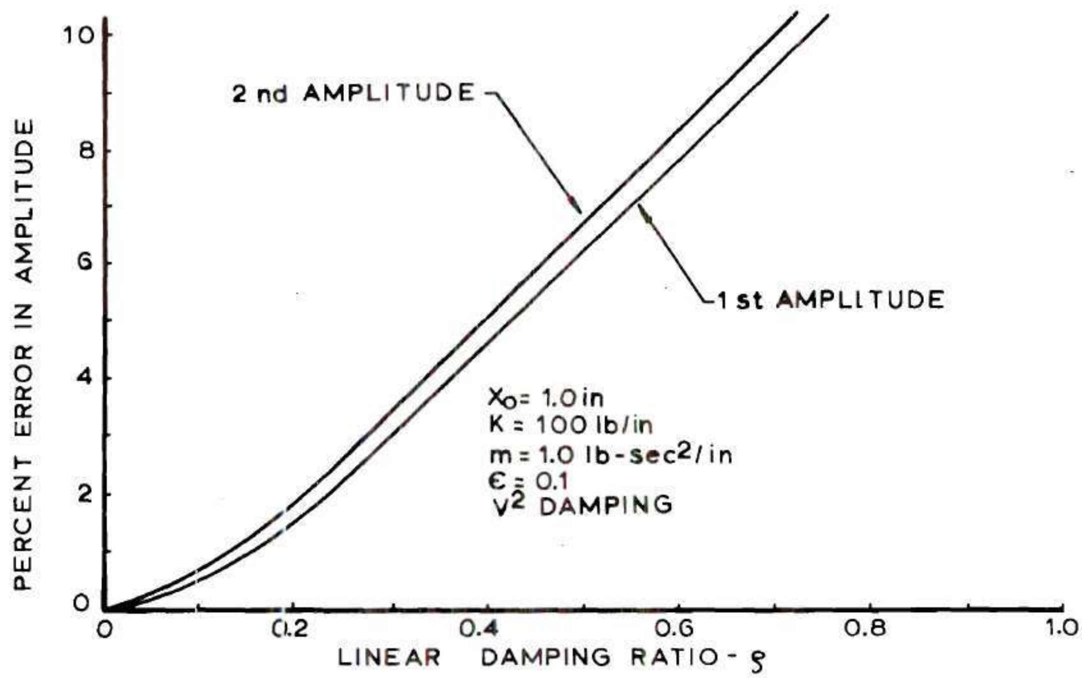


Figure 8. Error in Amplitude vs. Linear Damping Ratio--Extended K-B Approximation--$\zeta < 1$

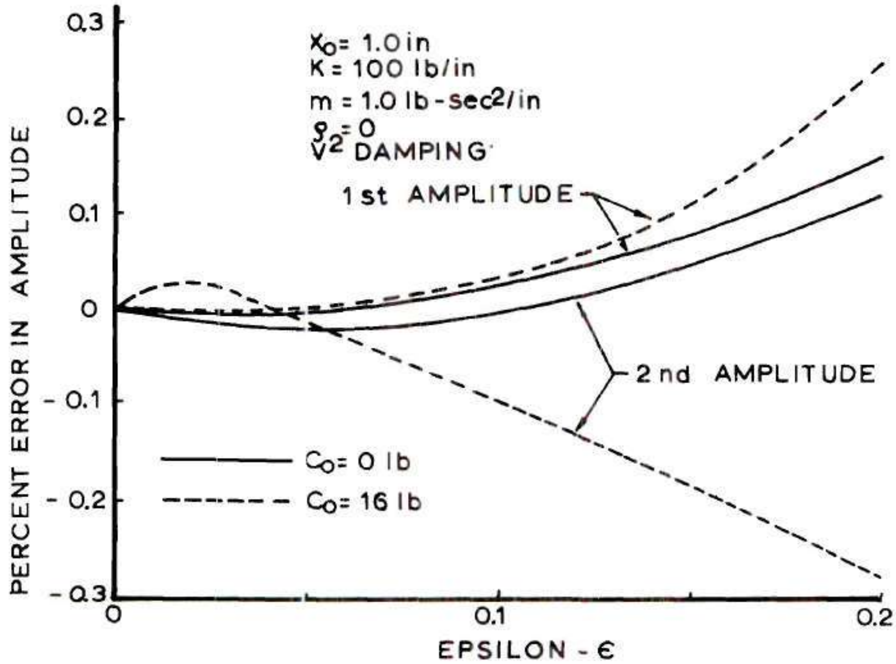


Figure 9. Error in Amplitude vs. Epsilon--Extended K-B Approximation--$\zeta = 0$

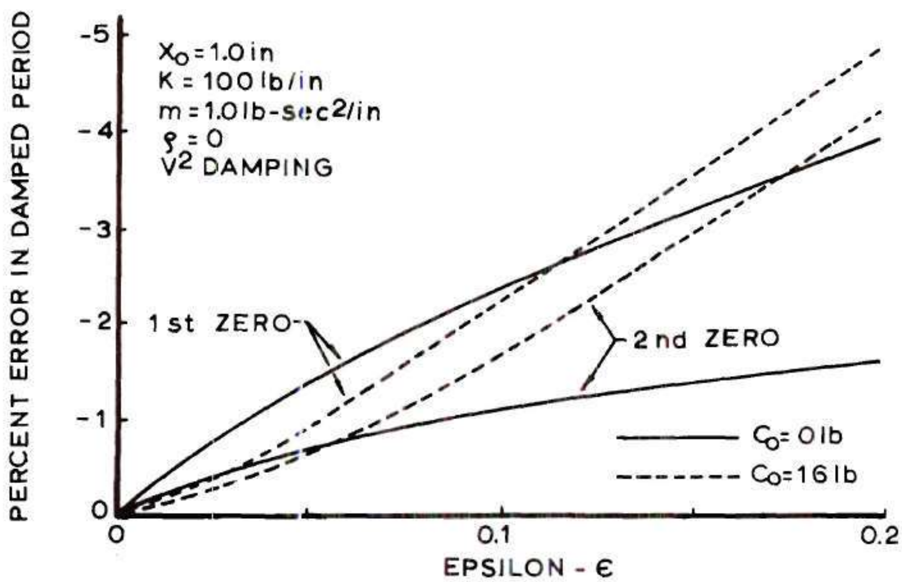


Figure 10. Error in Damped Period vs. Epsilon--Extended K-B Approximation--$\zeta = 0$

this small parameter raised to successively higher powers. The assumed series had the general form

$$x(t) = \phi_o(t) + \varepsilon\phi_1(t) + \varepsilon^2\phi_2(t) + \cdots \tag{2.14}$$

and thus

$$\dot{x}(t) = \dot{\phi}_o(t) + \varepsilon\dot{\phi}_1(t) + \varepsilon^2\dot{\phi}_2(t) + \cdots$$

and

$$\ddot{x}(t) = \ddot{\phi}_o(t) + \varepsilon\ddot{\phi}_1(t) + \varepsilon^2\ddot{\phi}_2(t) + \cdots$$

Here ε is a measure of the amount of nonlinear damping present in the system. Substituting the above expression into Equation (2.3) and collecting terms by powers of ε yields

$$\begin{aligned}
&\varepsilon^0[\ddot{\phi}_o + \frac{C_o}{m}\,\mathrm{Sgn}(\dot{x}) + 2\zeta\omega_n\dot{\phi}_o + \omega_n{}^2\phi_o] + \\
&\quad + \varepsilon^1[\ddot{\phi}_1 + 2\zeta\omega_n\dot{\phi}_1 + \omega_n{}^2\phi_1 + \gamma_2\mathrm{Sgn}(\dot{x})\dot{\phi}_o{}^2 + \gamma_3\dot{\phi}_o{}^3 + \cdots] + \\
&\quad + \varepsilon^2[\ddot{\phi}_2 + 2\zeta\omega_n\dot{\phi}_2 + \omega_n{}^2\phi_2 + 2\gamma_2\mathrm{Sgn}(\dot{x})\dot{\phi}_o\dot{\phi}_1 + 3\gamma_3\dot{\phi}_o{}^2\dot{\phi}_1 + \cdots] \\
&\quad + \cdots = 0
\end{aligned} \tag{2.15}$$

where ε and the γ_i's are defined in Equation (2.3). If the above are to be satisfied identically in ε, the coefficient of each power of ε

must vanish separately. This results in a series of differential equations which can be solved in a sequential manner.

The initial conditions that apply to this approximation are

$$\phi_o(o) = X_o \tag{2.16}$$

$$\phi_n(o) = 0 \quad \text{for } n = 1,2,3,\cdots$$

$$\dot{\phi}_n(o) = 0 \quad \text{for } n = 0,1,2,3,\cdots$$

These conditions are in agreement with the initial conditions of Equation (2.1).

Overcritical Viscous Damping

Mathematical Derivation. Considering first the coefficient to the zeroth power of ε in Equation (2.15), the following is the differential equation of the generating solution

$$\ddot{\phi}_o + \frac{C_o}{m}\,\mathrm{Sgn}(\dot{x}) + 2\zeta\omega_n\dot{\phi}_o + \omega_n^{\,2}\phi_o = 0 \tag{2.17}$$

For the overdamped case and with the initial conditions of Equation (2.16), the solution to the above is

$$\phi_o(t) = \frac{X^*}{\alpha - \beta}\left(\alpha e^{\beta t} - \beta e^{\alpha t}\right) + \frac{C_o}{m\omega_n^{\,2}} \tag{2.18}$$

$$\dot{\phi}_o(t) = \frac{\alpha\beta}{\alpha - \beta}\,X^*\left(e^{\beta t} - e^{\alpha t}\right)$$

where α and β are defined in Equation (2.5). Also, note that X^* is the amplitude of motion corrected for the presence of Coulomb friction.

The differential equation resulting from the coefficient of the first power of ε is

$$\ddot{\phi}_1 + 2\zeta\omega_n\dot{\phi}_1 + \omega_n{}^2\phi_1 + \gamma_2 \mathrm{Sgn}(\dot{x})\dot{\phi}_o{}^2 + \gamma_3\dot{\phi}_0^3 + \cdots = 0 \qquad (2.19)$$

The solution to this equation, with the initial conditions of Equation (2.16), results in the first order correction to the linear representation of the nonlinear problem. The details of solving this differential equation are given in Appendix B, and from these $\phi_1(t)$, for a general nth order damping law, can be expressed as

$$\phi_1(t) = \sum_{i,j=0,1,2\ldots}^{n} \frac{C_{ij}}{\alpha - \beta}\left[(\beta - j\alpha - i\beta)e^{\alpha t} - (\alpha - j\alpha - i\beta)e^{\beta t} + (\alpha-\beta)e^{(j\alpha+i\beta)t}\right] \qquad (2.20)$$

with the relationship $2 \le i + j \le n$ in effect and where

$$C_{ij} = \frac{(-1)^{j+i}\, d_{ij}\left(\frac{\alpha\beta X^*}{\alpha - \beta}\right)^{i+j}}{(j\alpha+i\beta)^2 + 2\zeta\omega_n(j\alpha+i\beta) + \omega_n{}^2}$$

and

$$d_{ij} = \begin{cases} D_{ij}\gamma_{i+j}\,\mathrm{Sgn}(\dot{x}) & \text{for } i+j = 2,4,6,\cdots \\ D_{ij}\gamma_{i+j} & \text{for } i+j = 3,5,\cdots \end{cases}$$

The values for D_{ij} are given in Table 8, Appendix B. In this relationship, the sum of i plus j must always be equal to one of the powers included in the nonlinear dissipation function. For example, with velocity squared damping, i plus j must always equal two.

Example of Application. The total response of a nonlinear system is expressed as a combination of Equations (2.18) and (2.20). This is written as

$$x(t) = \phi_o(t) + \varepsilon\phi_1(t)$$

A system with linear and velocity squared damping and one with linear and three higher order damping terms was analyzed with this approximation technique. These results are shown in Figures 11, 12, and 13. The first of these indicates the error for a system whose dissipation function includes a velocity squared term and one which includes three higher order terms. In this second case, the following relationship between the coefficients of the nonlinear terms held

$$C_4 = \frac{1}{2}C_3 = \frac{1}{2}C_2 \qquad (2.21)$$

Figure 12 and 13 are drawn for a system with velocity squared damping at a time corresponding to a displacement which is two-thirds of the initial displacement.

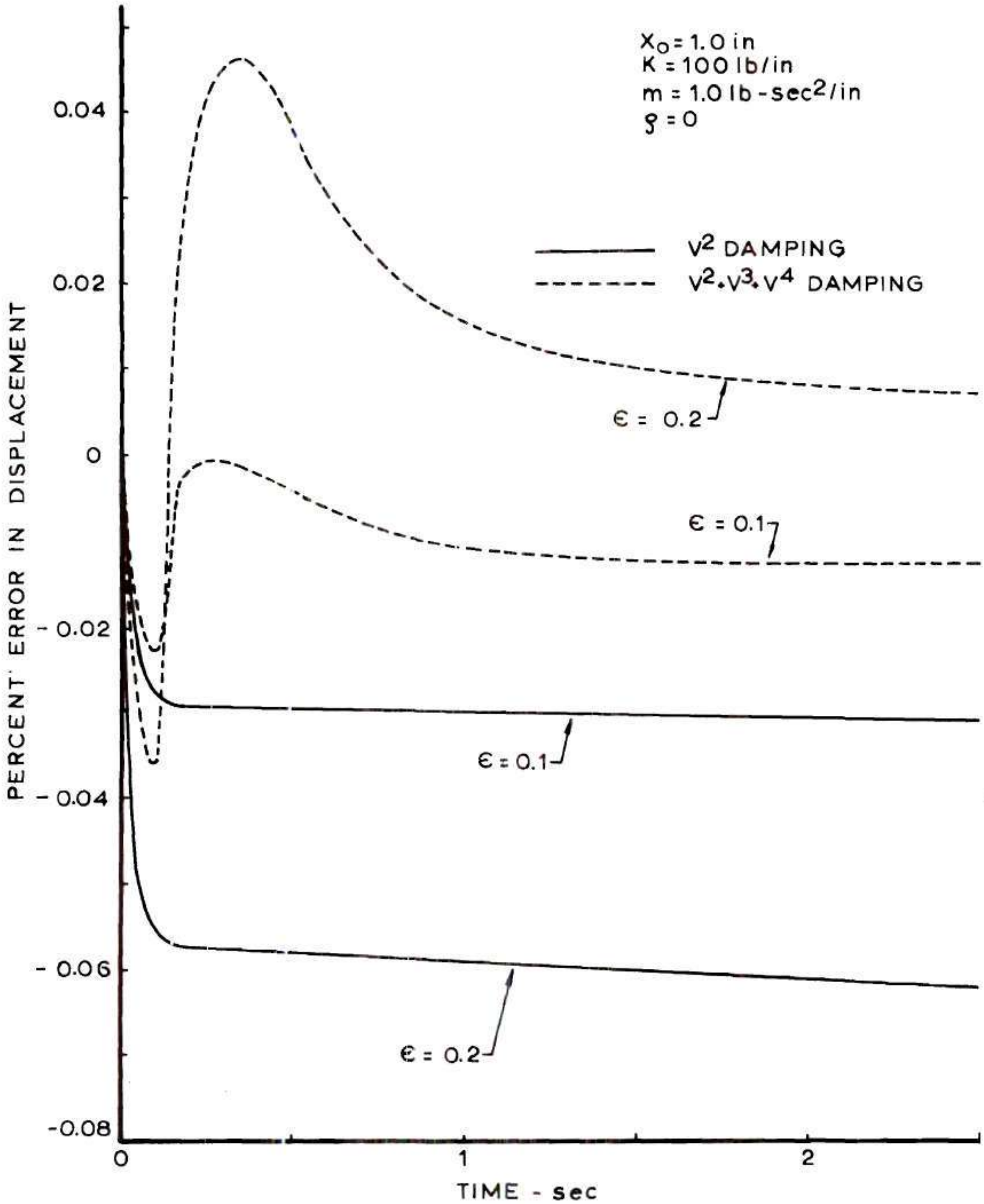


Figure 11. Error in Displacement vs. Time--
Perturbation Series Approximation--$\zeta > 1$

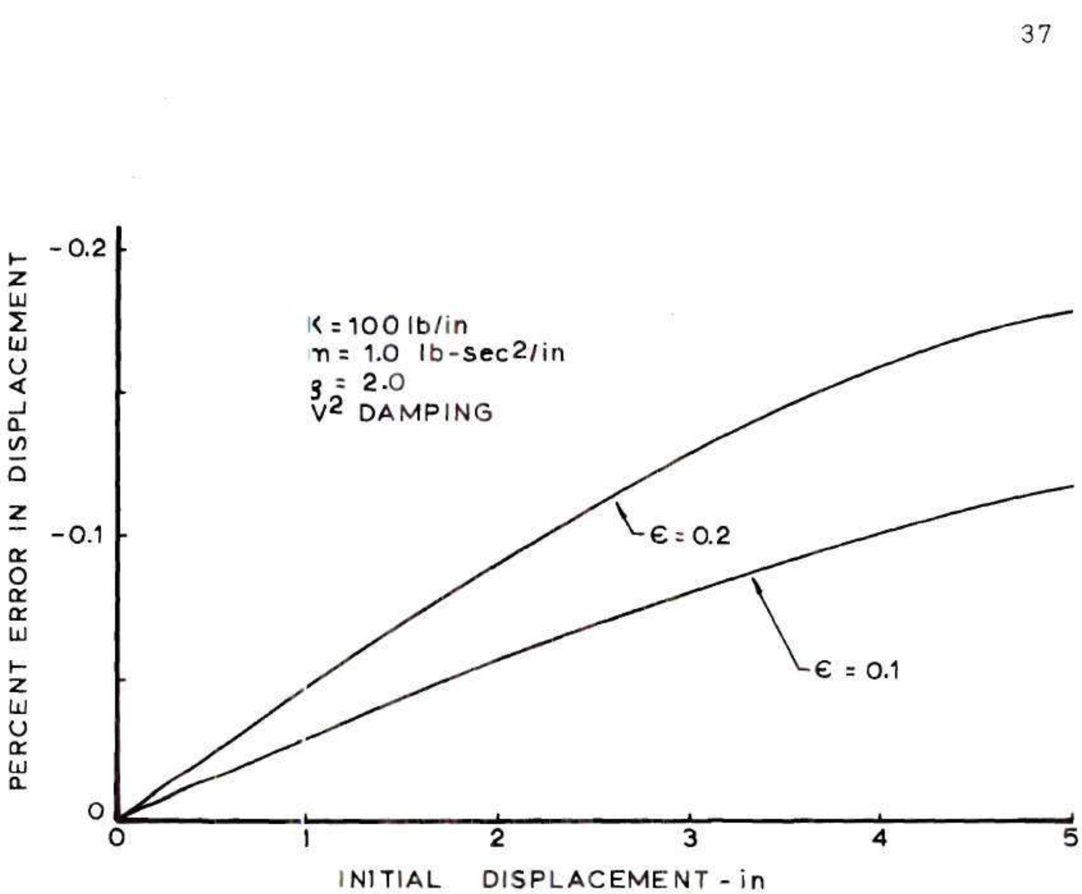


Figure 12. Error in Displacement vs. Initial Displacement--Perturbation Series Approximation--$\zeta > 1$

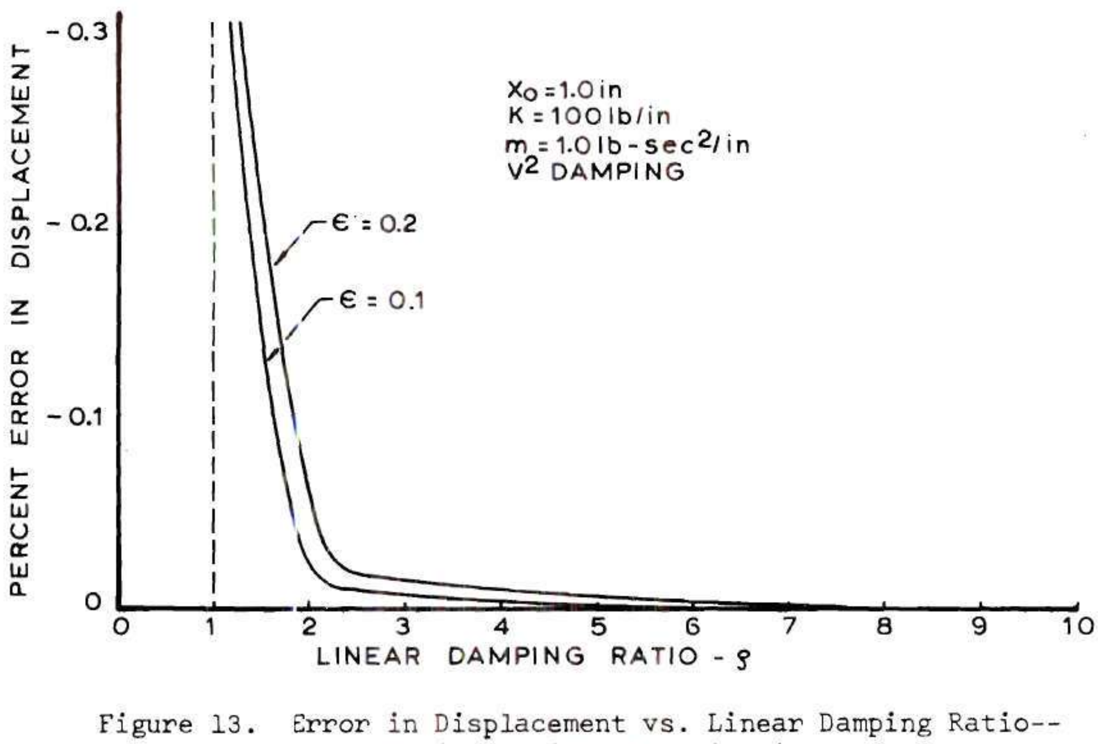


Figure 13. Error in Displacement vs. Linear Damping Ratio--Perturbation Series Approximation--$\zeta > 1$

Undercritical Viscous Damping

Mathematical Derivation. As was the case with overcritical viscous damping, Equation (2.17) is the governing equation for the zero order response $\phi_o(t)$. With the viscous damping term less than critical, this generating solution is

$$\phi_o(t) = X^* e^{-\zeta\omega_n t}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\,\text{Sin}\omega_d t + \text{Cos}\omega_d t\right] - \frac{C_o}{m\omega_n^2}\,\text{Sgn}(\dot{x}) \qquad (2.22)$$

$$\dot{\phi}_o(t) = -\frac{\omega_n X^*}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\,\text{Sin}\omega_d t$$

These expressions are consistent with the initial conditions given in Equation (2.16).

The expression for the first order correction term was again obtained from Equation (2.19), except that the linear damping ratio was limited to less than 1.0. There appeared to be no compact form for expressing this response as there was in the overdamped case and the resulting form of $\phi_1(t)$ was quite complicated in nature. Because of this, the first order correction has been obtained for a system whose dissipation function contains terms proportional up to and including the fourth power of the velocity. The steps required in obtaining this relationship for $\phi_1(t)$ are presented in Appendix B. From these considerations, $\phi_1(t)$ for a fourth order damping law becomes

$$\phi_1(t) = \frac{\gamma_2 \text{Sgn}(\dot{x})}{2(9-8\zeta^2)} \left(\frac{X^*}{\sqrt{1-\zeta^2}}\right)^2 e^{-\zeta\omega_n t} \{-4\zeta\sqrt{1-\zeta^2}\,\text{Sin}\omega_d t \tag{2.23}$$

$$+ 12(1-\zeta^2)\text{Cos}\omega_d t + e^{-\zeta\omega_n t}[-4\zeta\sqrt{1-\zeta^2}\,\text{Sin}2\omega_d t$$

$$+ (4\zeta^2-3)\text{Cos}2\omega_d t - (9-8\zeta^2)]\}$$

$$+ \frac{\omega_n \gamma_3}{16(4-3\zeta^2)} \left(\frac{X^*}{\sqrt{1-\zeta^2}}\right)^3 e^{-\zeta\omega_n t} \{6(1-\zeta^2)\text{Sin}\omega_d t$$

$$+ \frac{3\zeta(5-4\zeta^2)}{\sqrt{1-\zeta^2}} \text{Cos}\omega_d t + e^{-2\zeta\omega_n t}[3(4-3\zeta^2)\text{Sin}\omega_d t$$

$$- (3\zeta^2-2)\text{Sin}3\omega_d t + \frac{3\zeta(4-3\zeta^2)}{\sqrt{1-\zeta^2}} \text{Cos}\omega_d t + 3\zeta\sqrt{1-\zeta^2}\,\text{Cos}3\omega_d t]\}$$

$$+ \frac{\omega_n^2 \gamma_4 \text{Sgn}(\dot{x})}{24(1+8\zeta^2)(25-16\zeta^2)} \left(\frac{X^*}{\sqrt{1-\zeta^2}}\right)^4 e^{-\zeta\omega_n t} \cdot \{$$

$$\cdot \{\frac{4\zeta}{\sqrt{1-\zeta^2}} (8\zeta^4-4\zeta^2-15)\text{Sin}\omega_d t + 320(1-2\zeta^2+\zeta^4)\text{Cos}\omega_d t$$

$$- (25-16\zeta^2)e^{-3\zeta\omega_n t}[9+4(1-4\zeta^2)\text{Cos}2\omega_d t - 16\zeta\sqrt{1-\zeta^2}\,\text{Sin}2\omega_d t]$$

$$- (1+8\zeta^2)e^{-3\zeta\omega_n t}[(8\zeta^2-5)\text{Cos}4\omega_d t + \frac{4\zeta}{\sqrt{1-\zeta^2}} \text{Sin}4\omega_d t]\}$$

Example of Application. Combining Equations (2.22) and (2.23) in the following manner gives the corrected system response.

$$x(t) = \phi_o(t) + \varepsilon\phi_1(t)$$

The results of this approximation are shown in Figures 14 through 18. A system with a mass of 1.0 lb-sec^2/in and a spring rate of 100 lb/in has been considered. These curves are presented for a system with viscous plus velocity squared damping and one with viscous plus three higher order damping terms. In this second case, the relationship between the nonlinear coefficients was again governed by Equation (2.21). In these figures, the comparisons have been made for the displacements at the end of the first and second damped period.

Allowing the damping ratio ζ to go to zero in Equation (2.23) yields the response for a system with no viscous damping present. The results for this case, both with and without a Coulomb term present, are given in Figure 18.

Summary

Two techniques for constructing an approximate solution to Equation (1.1) have been outlined in the preceding sections. These consisted of a perturbation series approach and a modification of the classic Kryloff-Bogoliuboff method. Examples of the error associated with each of these methods were presented. The two methods offered here are an extension of the present approximations because no restriction has been placed on the magnitudes of the viscous and Coulomb coefficients. This allows both subsident and highly damped, oscillatory systems to be studied.

With the Perturbation Series Approximation, a system whose dissipation function is described by a nth order polynomial in the velocity can be analyzed. However, this technique resulted in quite a

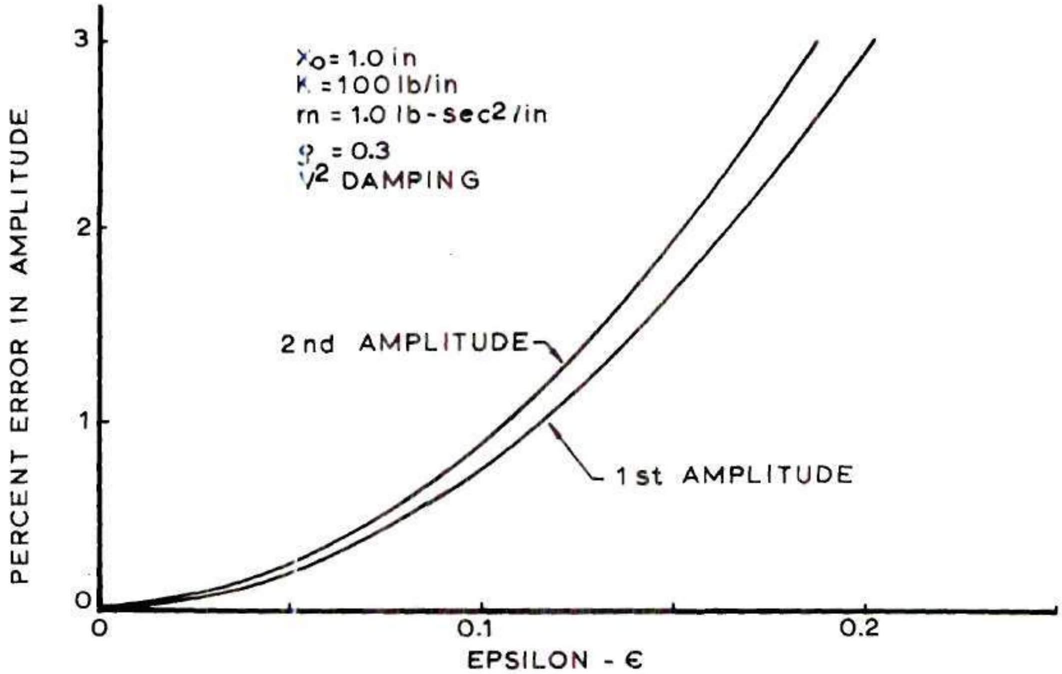


Figure 14. Error in Amplitude vs. Epsilon--Perturbation Series Approximation--$\zeta < 1$

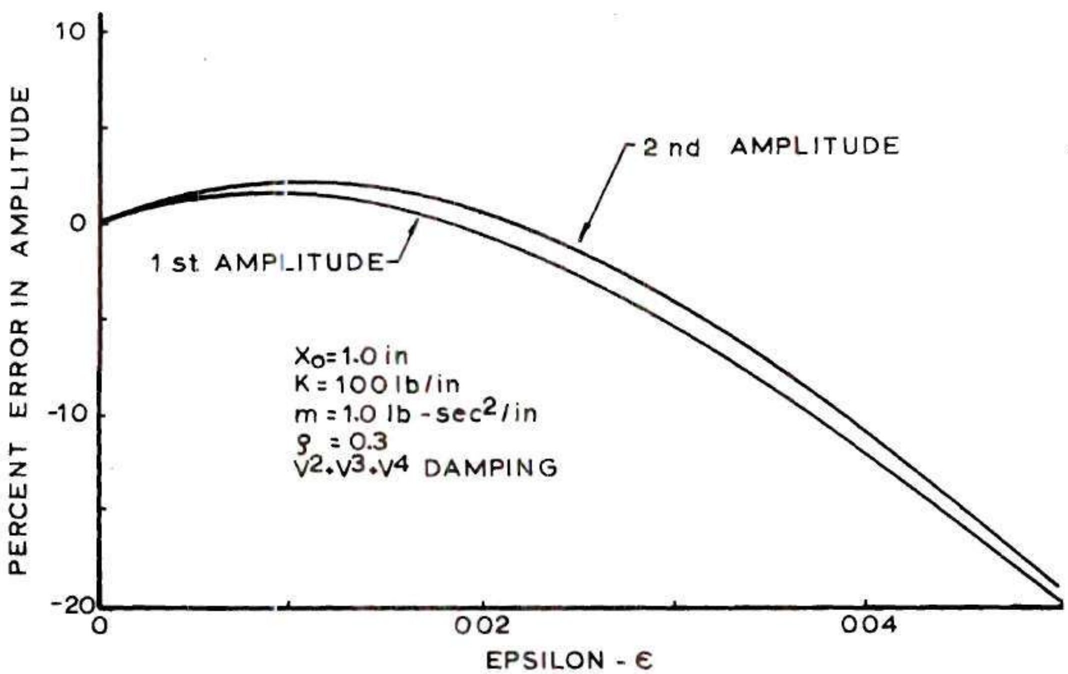


Figure 15. Error in Amplitude vs. Epsilon--Perturbation Series Approximation--$\zeta < 1$

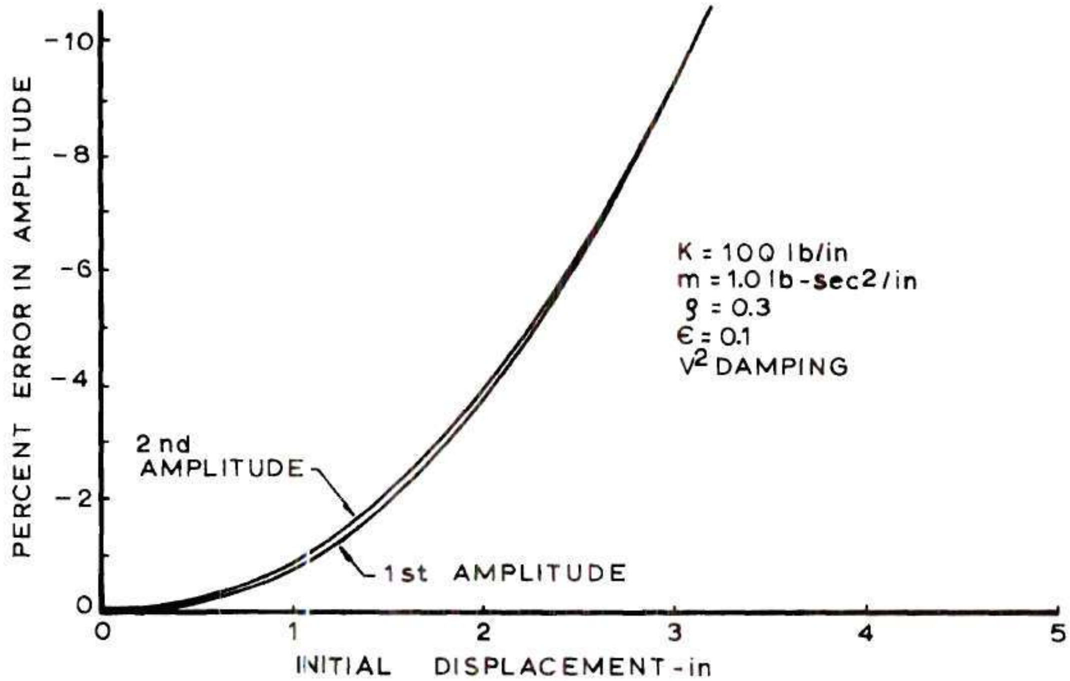


Figure 16. Error in Amplitude vs. Initial Displacement--Perturbation Series Approximation--ζ < 1

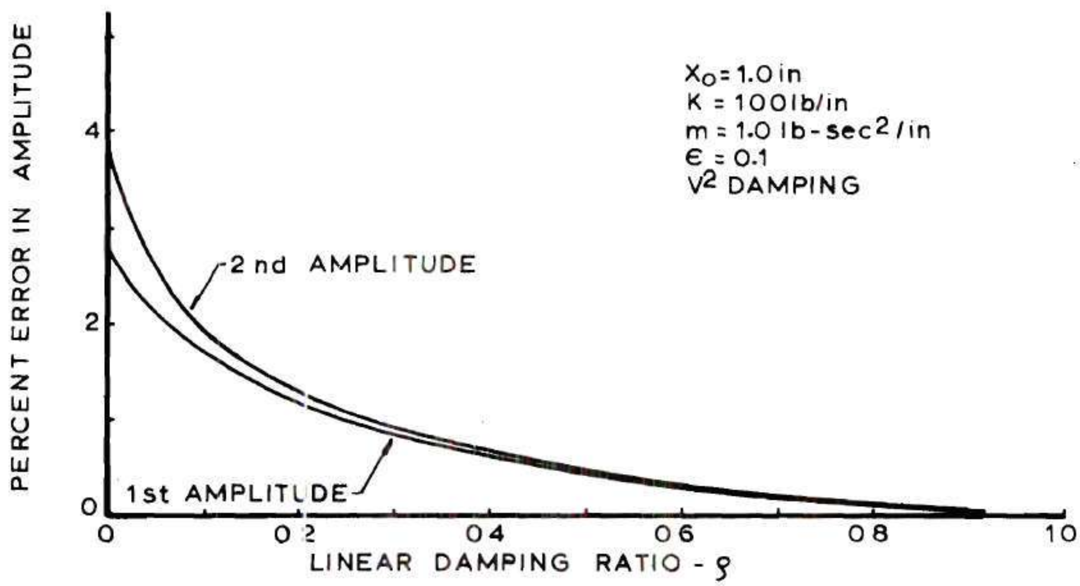


Figure 17. Error in Amplitude vs. Linear Damping Ratio--Perturbation Series Approximation--ζ < 1

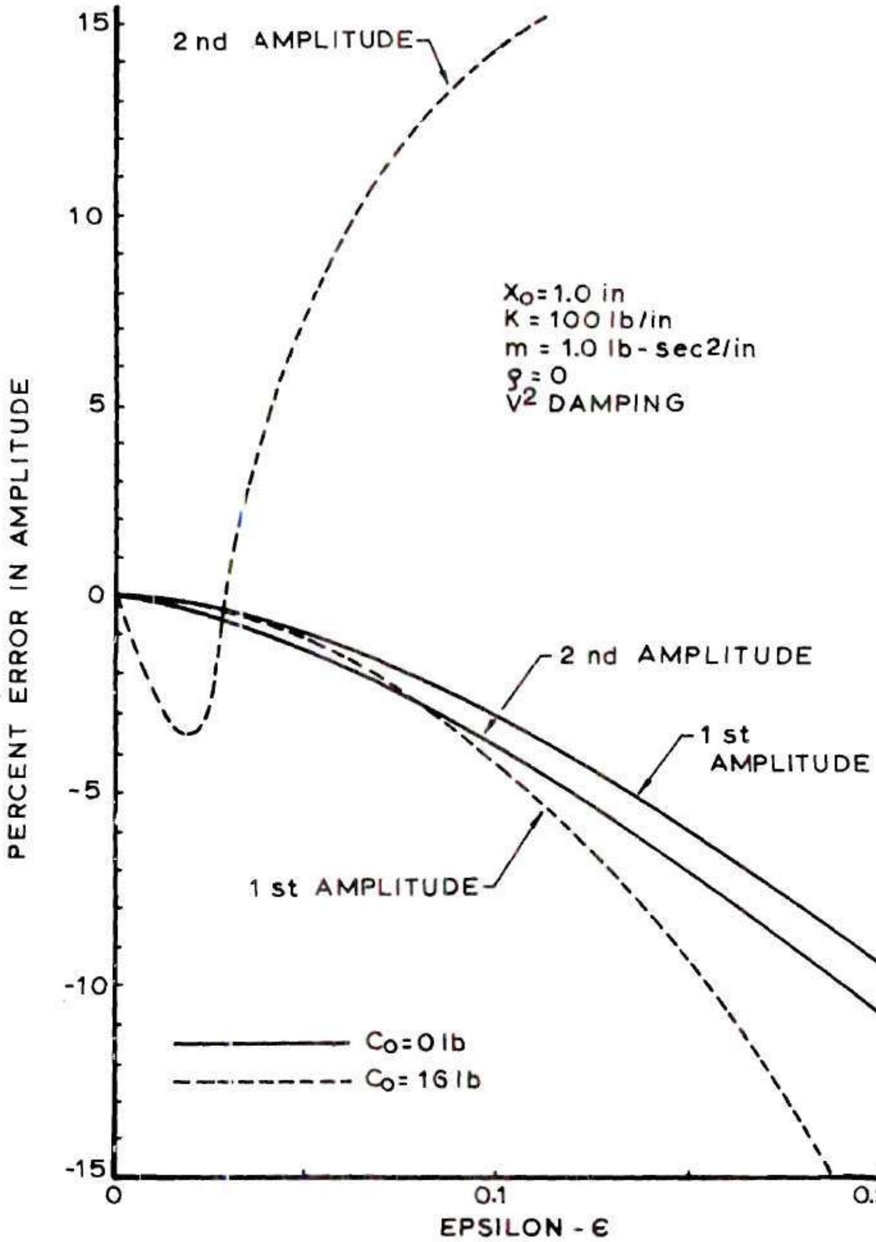


Figure 18. Error in Amplitude vs. Epsilon--Perturbation Series Approximation--ζ = 0

complicated representation of the system's response. It should also be noted that including additional terms in the dissipation function decreased the magnitude of nonlinearity that could be handled by this approximation. The Extended K-B Approximation yielded a simpler response expression, but its application was limited to a system with Coulomb and viscous damping plus one higher order term.

Quite often with perturbation series methods, the problem of secular terms is encountered. At no point in the analysis performed here, did this question appear. However, if the viscous damping coefficient was removed before the formulation of the approximate solution, in certain cases it might be possible to obtain secular terms. The consideration of secular terms is discussed more completely in Appendix C.

From the comparisons which have been made, it can be seen that the accuracy of the approximations are affected by not only the amount of nonlinearity present, but also the magnitudes of the initial displacement and the linear damping coefficient. When the linear damping term was greater than critical, the perturbation series approach gave a much better approximation to the response. In the case of the damping ratio being less than critical, the Extended K-B Approximation seemed to be more accurate.

CHAPTER III

EXPERIMENTAL INVESTIGATION

Experimental Apparatus

A damped spring-mass model was constructed with which the transient response histories for various damped physical systems were obtained. This device was built in such a manner that tests could be conducted with various moving masses and spring rates in conjunction with different types of damping.

A sketch of the equipment is shown in Figure 19. The mass of the system was supported by linear ball bushings which rode on two polished steel shafts. These shafts supported the mass and also acted as guides for the linear bearings. Additional weights could be added to the basic mass in order to increase the total weight of the system. Two springs were used to provide the linear restoring force. These springs were attached between the mass and the two rigid supports at the ends of the guide shafts. The spring rate of the system could easily be altered by interchanging these springs.

The damping force was provided by a fin which was located on the lower side of the mass. This fin was immersed in a free-surface oil bath. Fins with different geometric properties and various orientations with respect to the direction of motion could be provided. Also, different weight oil was used with various springs in order that both oscillatory and subsident motion could be studied.

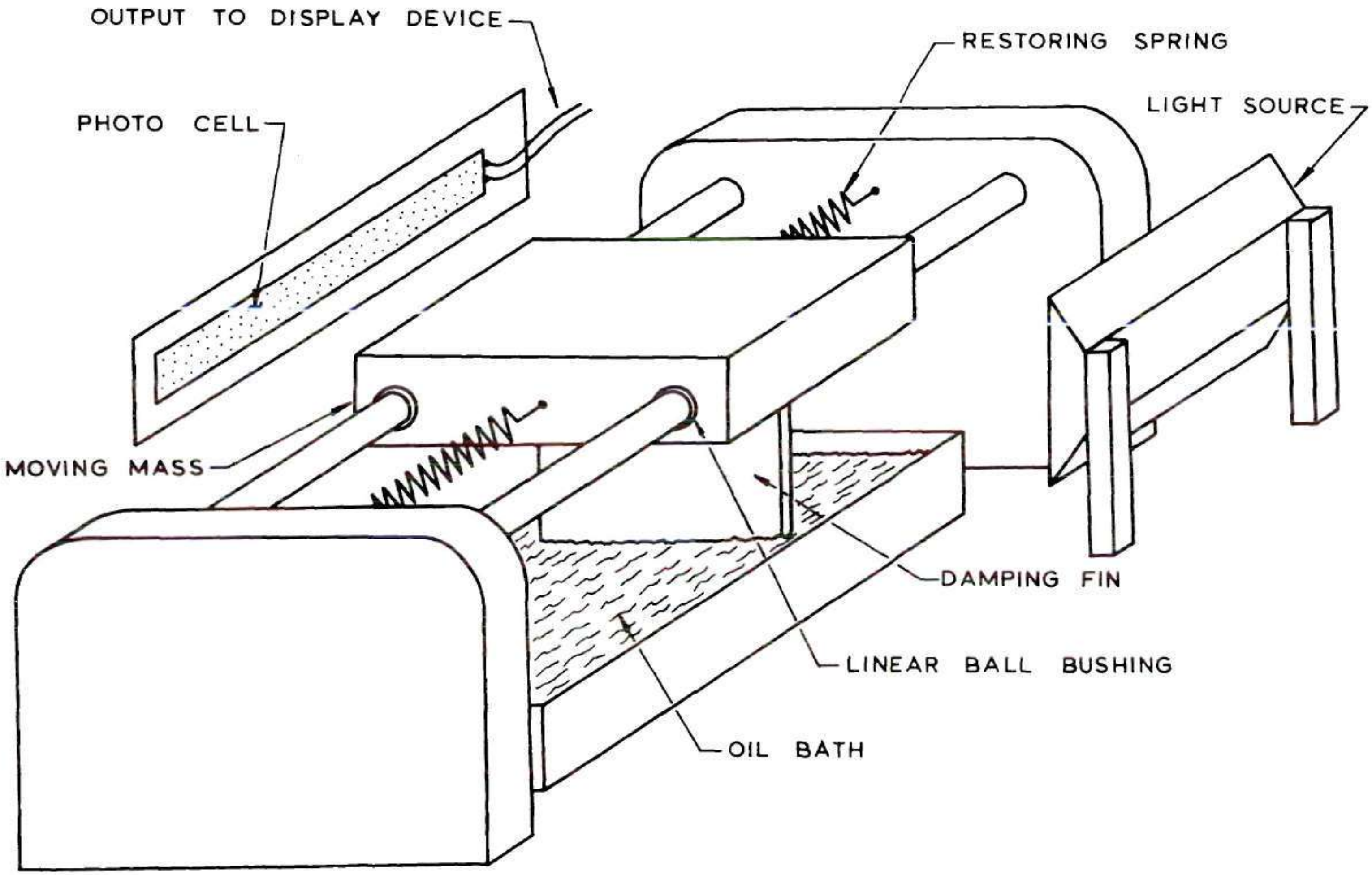


Figure 19. Experimental Apparatus

Displacement Transducer

A light sensitive transducer was used to obtain the displacement-time relationship for the dynamic system. A selenium photovoltaic cell was attached to a stationary support and illuminated by a constant light source. The moving body was located between the light and the photo cell, Figure 19, and thereby cast a shadow on the cell. As the system and its shadow were displaced, the illuminated area of the photo cell changed and the resulting voltage change was related to the displacement of the system.

The photo cell that was used was essentially a P-N type transistor (67) which produced an electrical voltage when illuminated. For a stronger light intensity or a larger illuminated area, the voltage output of the cell was higher. Figure 20 shows the circuit which was employed with the photo cell. The battery provided a reverse bias across the photo cell which improved both its linearity and the magnitude of its output (68). The output of this transducer circuit was displayed on an oscilloscope.

A string of four light bulbs provided the required illumination and a frosted lens was used to smooth out the light's intensity and eliminate any bright spots. These lights were powered with a DC voltage source and all the tests were conducted in a darkened room to insure a constant illumination.

To take advantage of the linearity of the photo cell's voltage output, the following procedure was followed before the cell was calibrated.

1. The entire area of the photo cell was illuminated with the desired light intensity.

2. The gain resistance, Figure 20, was reduced until the output voltage was zero.

3. The gain resistance was lowered slightly further to move away from the point of zero output voltage.

4. Care was taken not to overdo this last step as it tended to reduce the photo cell's output voltage.

The transducer was calibrated by measuring the static displacement of the mass with a pair of vernier calipers. A calibration curve was obtained by plotting this displacement versus the corresponding voltage output from the photo cell. As can be seen in Figure 21, this voltage-displacement relationship was linear over a large range. This particular calibration curve was used in conjunction with the subsident tests discussed in the next section.

To determine the frequency response of these photo cells, a shadow device, mounted on an electrodynamic shaker, was placed between a light source and a photo cell. The transmissibility of this arrangement for frequencies up to 100 cps was measured. It was found that the photo cell's transmissibility was flat in this frequency range.

Experimental Results

The results from tests conducted on two different systems are presented here. A system whose motion was subsident in nature and one with oscillatory motion are included. The configurations of these dynamic systems are given in Table 1. Typical oscilloscope records of

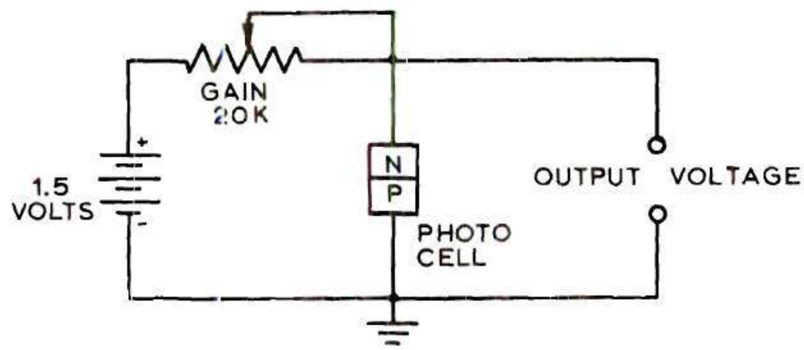


Figure 20. Photo Cell Displacement Transducer Circuit

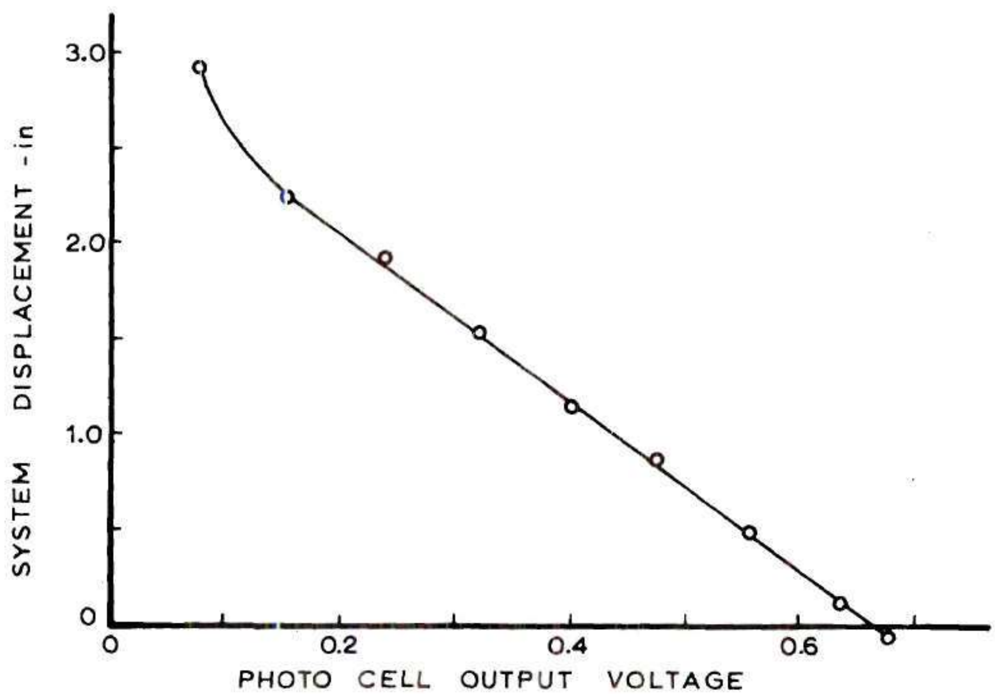


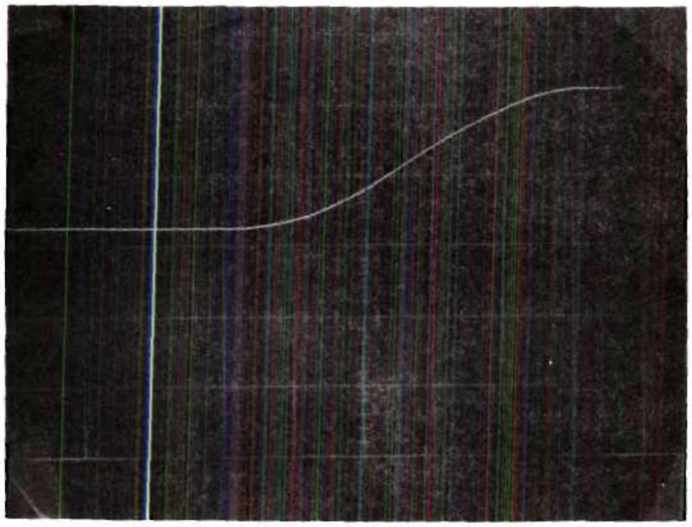
Figure 21. Typical Photo Cell Calibration Curve

the response obtained for these two configurations are shown in Figure 22.

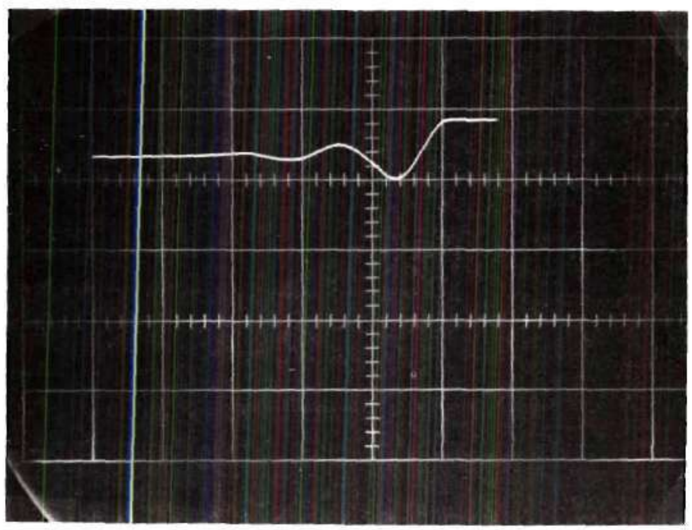
For each test case, time histories were obtained for several different initial conditions. This information was required for the final evaluation of the results of the searching study. In each case, the mass was brought back against a rigid stop and then released. A micro switch was located on this stop and it triggered the oscilloscope trace when the mass was released.

Table 1. Properties of Experimental Systems

System	Subsident	Oscillatory
Mass ($lb\text{-}sec^2/in$)	.002137	.001400
Spring Rate (lb/in)	.05042	.09623
Fin Orientation	Skew	Perpendicular
Damping Fluid	SAE 40 Oil	None



Subsident System



Oscillatory System

Figure 22. Typical Oscilloscope Records

CHAPTER IV

OPTIMUM SEARCH PROCEDURES

Introduction

A computational algorithm has been developed which yields the coefficients of the dissipation function for a single degree of freedom dynamic system whose displacement-time history is known. This routine employs an optimization technique for determining these unknown parameters. Two different descriptions of the system's response were used with this optimum search routine. The first of these, referred to as the Approximate Solution Criterion Function, was applied to a system for which an approximate, closed form expression of the response existed. This procedure was limited in its area of application by the approximate expression that was used and the inherent limitations of the approximation. The second approach, called the Numerical Integration Criterion Function, was quite general in nature. It could be applied to a system whose motion is either oscillatory or subsident and whose dissipation function was described by a general nth order polynomial in the system's velocity. In this procedure, the response of the system was obtained by numerical integration of the equation of motion.

In each of these searching procedures, a criterion function is established which measures the agreement between the system's analytical and measured response. A classical least squares criterion function was used throughout this investigation. The form of this criterion function can be expressed as

$$CF = \sum_{i=1}^{N} [x_m(t_i) - x_a(t_i)]^2 \tag{4.1}$$

where

x_m = measured response.

x_a = analytical response.

N = number of measurements.

t_i = time of ith measurement.

This form of the criterion function will tend to supress small positive and negative errors which might be present in the observed data. In addition, with this criterion function, a great many points from the system's displacement-time history can be included in the comparison with the mathematical response.

The general procedure in a search routine is to evaluate the criterion function for a particular set of independent variables, in this case the damping coefficients. This procedure can be repeated many times over the area of interest until an extremum point has been located. Instead of just bracketing the area of interest and saturating it with trials, the value of each new set of assumed damping coefficients can be influenced by information gained from the previous calculations. This more efficient search procedure is what is known as an optimum search technique.

Multidimensional Search Technique

A computer routine has been developed, with which the coefficients of a dissipation function represented by an nth order polynomial in velocity, can be determined. This procedure requires knowledge of the systems displacement-time history and its mass and spring rate. The response of the system can be represented by either a numerical integration technique or by some approximation method.

Both the Burroughs B-5500 and the Univac 1108 computers were used during this investigation. Since the programs for these two machines are essentially identical, only the ALGOL program for the Univac 1108 is presented in Appendix D. This program is set up so that the criterion function is calculated in a subroutine or procedure. In this way, the form of the criterion function can be varied without disrupting the main program. In addition, this routine can evaluate the coefficients of a dissipation function containing any combination of terms up to and including ones proportional to the fourth power of the velocity.

Outline of Search Routine

To avoid being too abstract, the search routine to be discussed will be limited to the case of two independent variables. Everything that will be developed can be directly extended to a multidimensional search problem as was done in the program given in Appendix D. Figures 23 and 24 present the geometric properties of the search routine, where C_i and C_j are the two independent variables and CF is the criterion function.

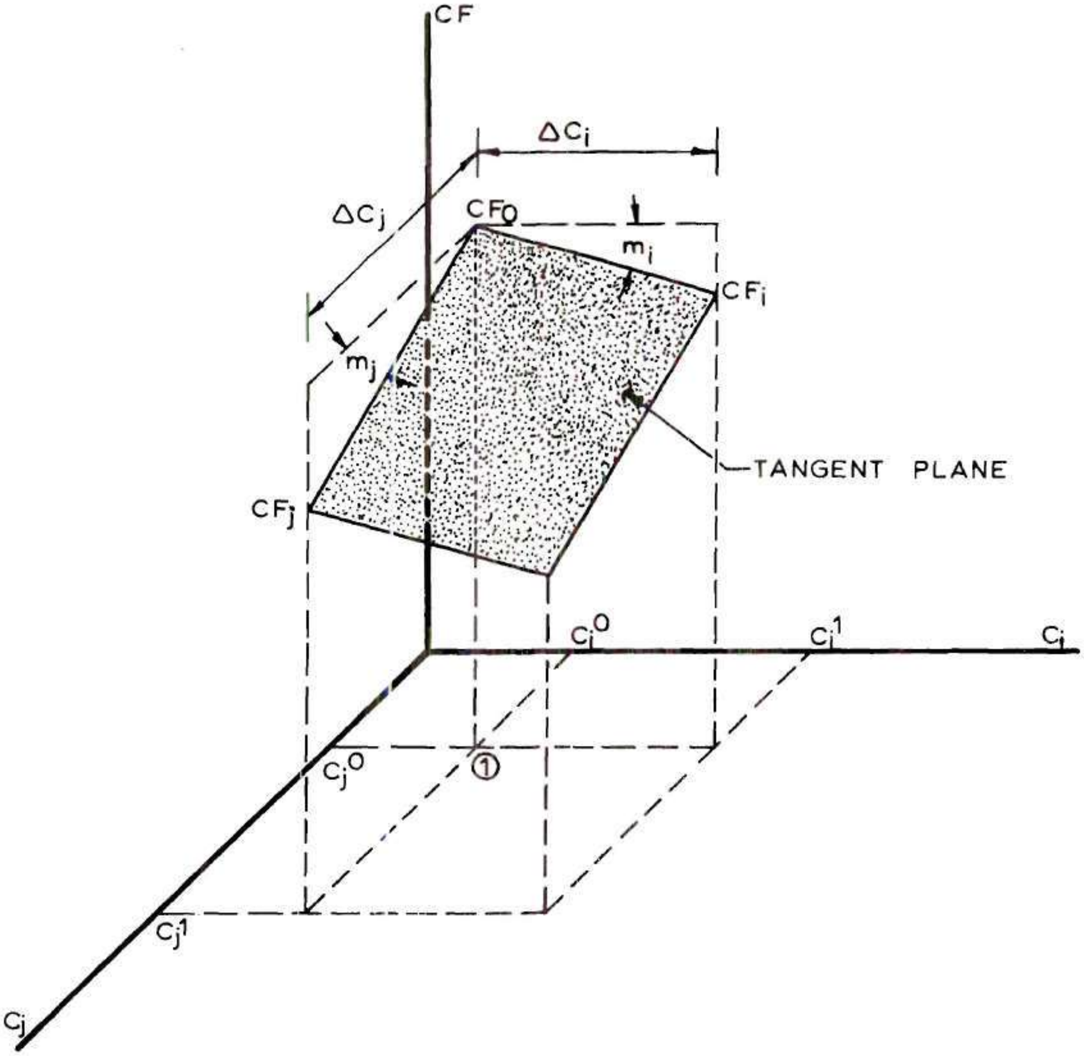


Figure 23. Geometry of the Criterion Function Response Surface

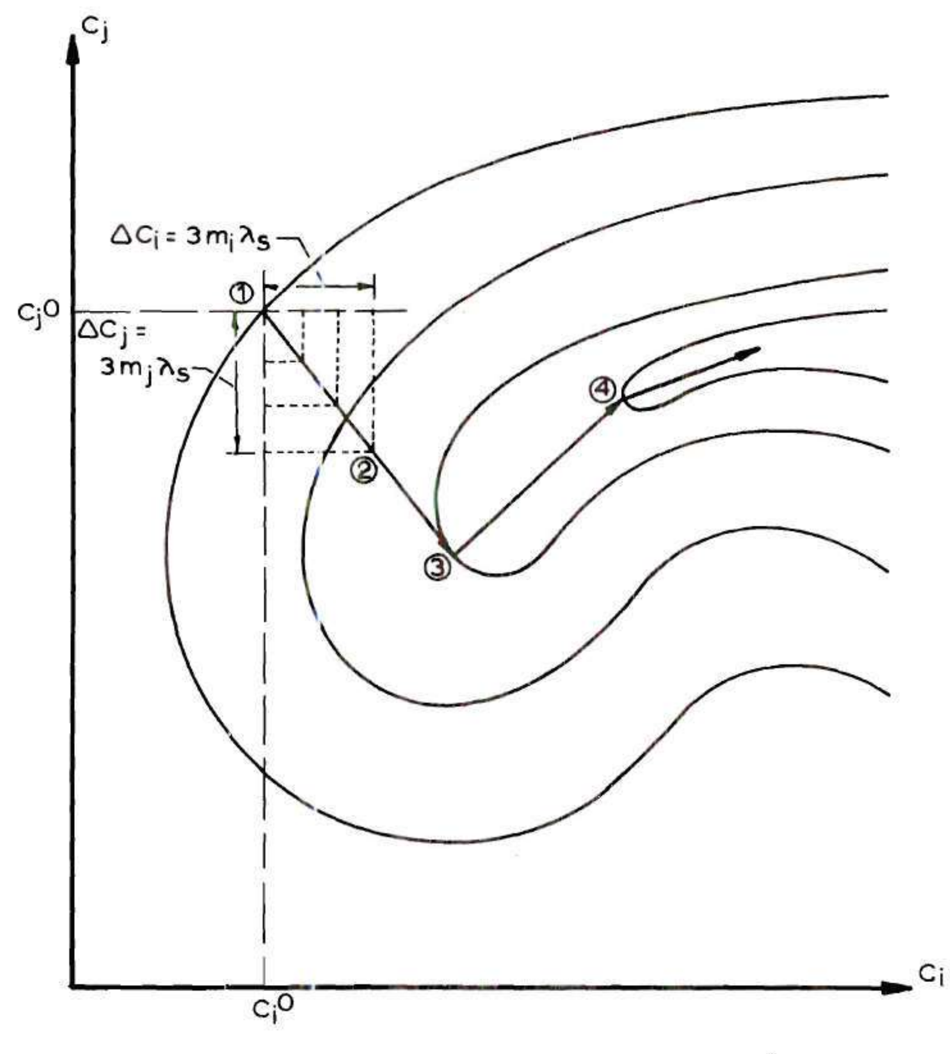


Figure 24. Projection of Criterion Function on C_i-C_j Plane

At this point, no assumption as to the exact nature of the criterion function is required, only that in some manner it measures the agreement between the system's measured response and the mathematically predicted response. In addition, the following discussion applies to either an approximate or numerical solution to the equation of motion.

Basically, the search technique that is used is a ramification of the method of steepest descent. An initial estimate of the values of the damping coefficients has to be made. The location of this initial estimate is shown as point 1 in Figures 23 and 24. Then the slopes, at this initial point, of the lines approximately tangent to the response surface in the directions of the independent variables are evaluated by

$$m_i = \left.\frac{\partial CF}{\partial C_i}\right|_o \approx \frac{CF_i - CF_o}{C_i^{\,1} - C_i^{\,0}} \tag{4.2}$$

$$m_j = \left.\frac{\partial CF}{\partial C_j}\right|_o \approx \frac{CF_j - CF_o}{C_j^{\,1} - C_j^{\,0}}$$

The equation of the plane approximately tangent to the response surface at this initial point is given as

$$CF = CF_o + m_i \Delta C_i + m_j \Delta C_j$$

In order to move down this plane in the direction of the steepest slope, the damping coefficients are varied in proportion to their respective slopes. This is expressed as

$$\Delta C_i = -\lambda m_i \quad \text{and} \quad \Delta C_j = -\lambda m_j \tag{4.3}$$

where λ is the proportionality constant and is the same for each of the ΔC's. The above relationships reduce the multidimensional problem to a single variable search on the stepping parameter λ.

The damping coefficients are varied according to Equation (4.3) for increasing values of λ and the corresponding criterion function is evaluated. The value of λ for the nth step down the response surface is determined by

$$\lambda = n\lambda_s \tag{4.4}$$

where λ_s is the initial value of λ and is set with the input data. For example, referring to Figure 24, the search would have moved to point 2 when n equals three in the above expression. This procedure is continued until the new value of CF is greater than the last value. If these two quantities differ by more than a prescribed amount, the search moves back down the gradient direction until the two neighboring values of the criterion function straddling an apparent minimum have reached an acceptable measure of closeness.

At this point in the search, a Golden Section routine (Wilde (64), p. 32 and Carnahan and Wilkes (65), p. 8-13) is used to locate the low point indicated as point 3 in Figure 24. The Golden Section approach is a single variable search technique which greatly diminishes the computation required to reduce the interval of uncertainty of the stepping parameter λ.

Using the notation of the program given in Appendix D, Figure 25 indicates the properties of this method.

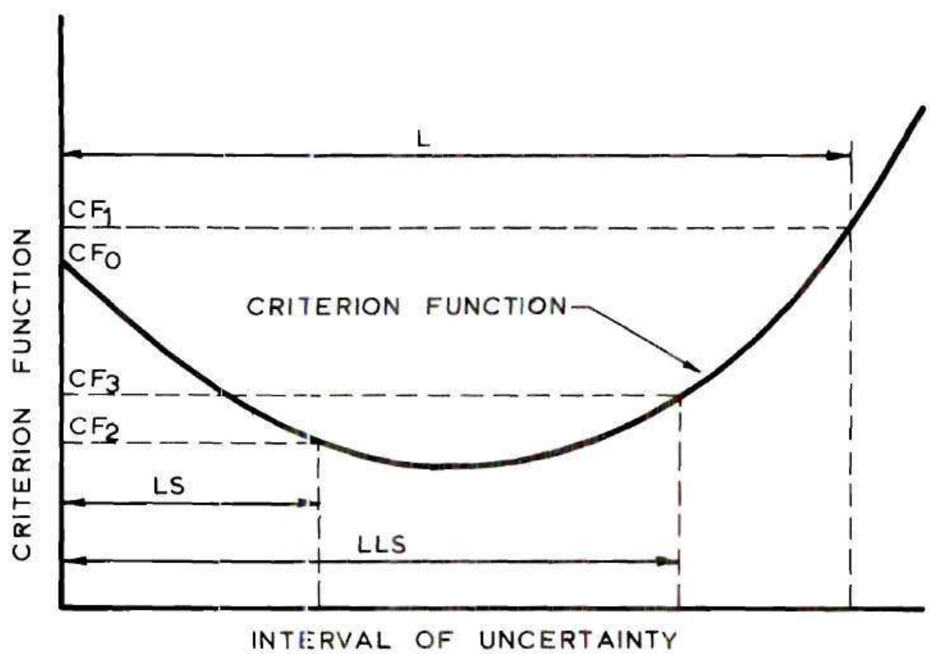


Figure 25. Golden-Section Search Routine

Here L is the length of the interval of uncertainty when control of the program is first switched to the Golden Section routine. Two additional experiments are performed within this interval at the points defined by

$$LS = (1-\tau)L = .382L$$

$$LLS = \tau L = .618L$$

In the above, τ is the Golden Section number, and is the ratio of successive interval lengths which has a value of 0.618. From this point on, it can be seen that only one additional evaluation of the criterion function is required for at least a 38.2 per cent reduction in the interval of uncertainty. This procedure is repeated until the interval is reduced to 1 per cent of its original length.

The slopes at this new location on the response surface, point 3 in Figure 24, are then calculated by Equation (4.2) and their magnitudes are compared with a convergence factor. If the value of all of the slopes are equal to or less than this tolerance, the search is terminated. At this point, the final value of the criterion function, the corresponding damping coefficients, and the system's response are tabulated. The measure of convergence is governed by the term FACTOR which is set with the input data.

If the values of all of the slopes do not satisfy the convergence test, the results from the Golden Section routine are stored and control of the program is transferred back to the beginning. The process of moving down the response surface, performing the Golden Section routine, and checking on convergence is repeated. This second step is shown as a movement from points 3 to 4 in Figure 24. This procedure is repeated until the convergence test is met or the value of the criterion function does not change after five consecutive trials. In this latter case, a note as to the lack of required convergence is made and the results of the search to that point are tabulated.

A discussion of the application of this search routine is delayed to a later chapter. Results are shown for both a physical system and a mathematically generated system.

Evaluation of the Criterion Function

Two methods were used to evaluate the least squares criterion function given in Equation (4.1). The first of these used one of the approximate solutions for the description of the system's response. The second, generated the response through a numerical integration routine. These criterion function evaluations were programmed as separate subroutines or procedures which could be changed without disturbing the main search program.

Approximate Solution Criterion Function. The evaluation of the criterion function in this case employs the results of the Extended K-B Approximation. The details of this approximation technique are given in Chapter II and a computer listing for this procedure is given in Appendix D.

The advantage of this technique is that the mathematical response has to be evaluated at only the times where comparison with the measured response is to be made. This greatly reduces the amount of computer time required by the search routine. The disadvantage is that the analytical response is at best an approximation to the actual response for the particular set of damping coefficients.

Numerical Integration Criterion Function. In this case, the criterion function is evaluated by a numerical integration technique. A fourth order, Runge-Kutta method (69) is used and a listing of this procedure is given in the third part of Appendix D.

This approach has the advantage that any nonlinear dissipation function can be analyzed and there exists no limitations on the magnitudes of the various damping coefficients. On the other hand, this procedure requires more computer time because the numerical integration routine has to move through the time interval of interest in small steps.

CHAPTER V

APPLICATION OF THE OPTIMUM SEARCH PROCEDURES

The Optimum Search Procedures discussed in the preceding chapter have been used to determine the dissipation function for several different dynamic systems. These procedures have been applied to transient response information obtained from a physical system and also to mathematically generated response data. In this latter case, the response of a system with a known dissipation function was numerically obtained, and this response was treated as the measured or observed data. Application of these techniques to both subsident and oscillatory systems are presented here.

Mathematically Generated Response Data

Limited Dissipation Function

Noise Free Observations. In order to gain some insight into the application of the developed procedures, a system with a known dissipation function has been considered. A system with a nonlinear dissipation function composed of a combination of Coulomb and velocity squared damping was studied. The Numerical Integration Criterion Function was used throughout the analysis conducted with this response information.

The equation of motion for the system considered was assumed to be of the form

$$1.0\ddot{x} + (20+3.5\dot{x}^2)\mathrm{Sgn}(\dot{x}) + 100x = 0 \tag{5.1}$$

with an initial displacement of one inch. The response of this configuration was obtained with a Runge-Kutta numerical integration procedure. The same procedure was used that was mentioned in the discussion of the numerical criterion function in the preceding chapter. It should be noted that the response of this assumed system with the assumed initial displacement is subsident in nature.

During the initial stages of the analysis, the search was performed for a dynamic system having the same form as Equation (5.1). Here, only C_0 and C_2 were varied in a system described by

$$1.0\ddot{x} + (C_0+C_2\dot{x}^2)\mathrm{Sgn}(\dot{x}) + 100x = 0$$

The search procedure was performed for various initial estimates of the coefficients C_0 and C_2. The results of this phase of the analysis are displayed in Figure 26 as contours of the criterion function response surface. Regardless of the location of the search's initial point on the C_0-C_2 plane, the search always moved toward an area about the true dissipation function. All of these initial trials resulted in a system description falling within the shaded region shown in Figure 26. When the tolerance of convergence within the shaded area was tightened, improved agreement with the true damping law was obtained. Without undue computational effort, a value of C_0 of 19.54 lbs and a value of C_2 of 3.599 lb-sec^2/in^2 was obtained. Both of these are within 3.0 per cent of the assumed damping coefficients given in Equation (5.1).

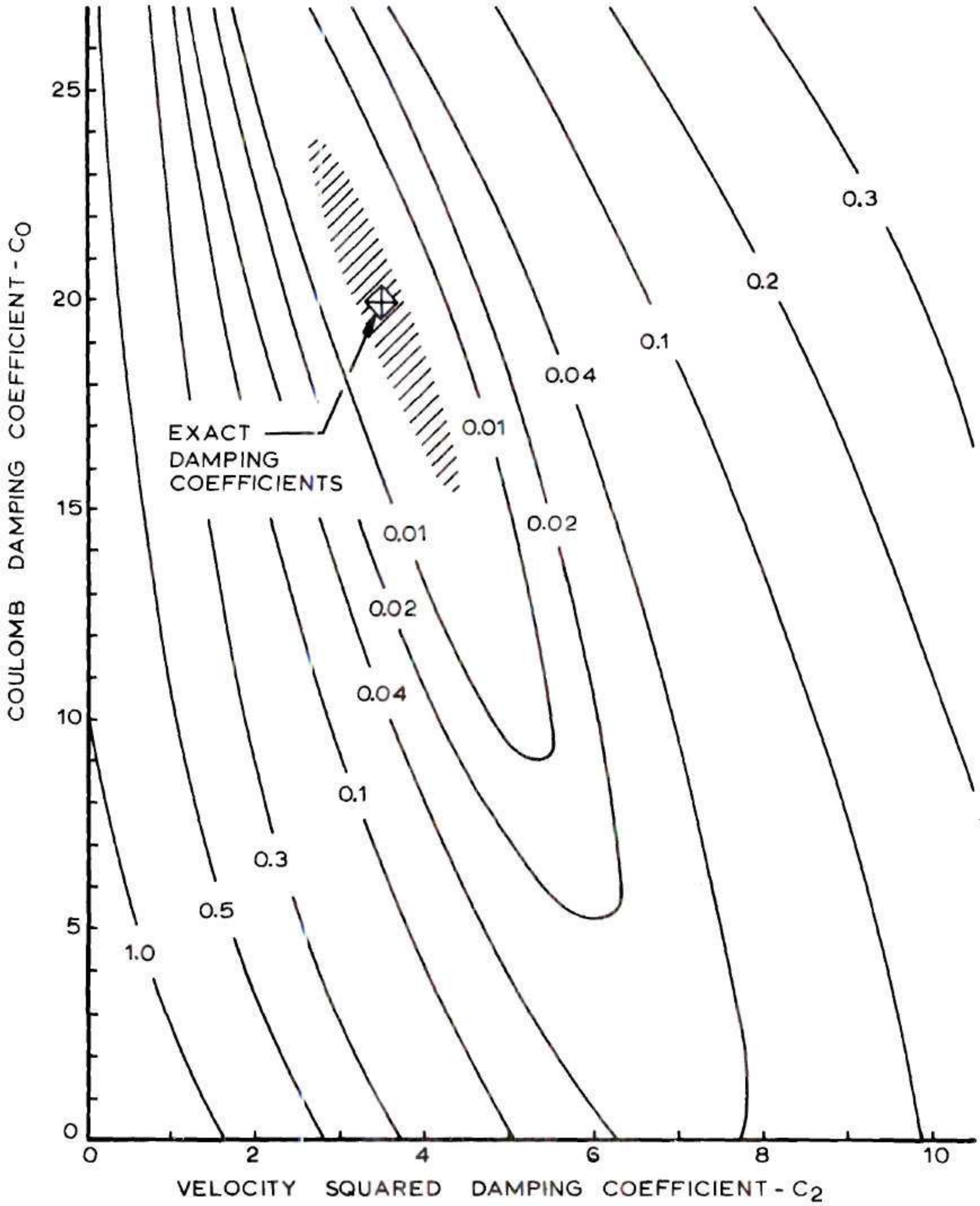


Figure 26. Contours of Criterion Function Response Surface

As can be seen in Figure 26, the contours of the response surface were quite elliptical in nature, Wilde (64), p. 119, points out that gradient search methods are most efficient if the contours are circular in nature. For this reason, the search procedure was scaled to provide contours which were more nearly of this form. This scaling was implemented through the array SFACTOR given in the listing in Appendix D. The new variables in the search were the scaled coefficients, CS, where these quantities were expressed as

$$CS_i = SFACTOR_i \cdot C_i$$

In this investigation, it was found that the following array of scale factors gave good results for all of the systems considered.

$$SFACTOR_i = 1,1,10,50,100 \quad \text{for } i = 0,1,2,3,4$$

<u>Noisy Observations</u>. Next, values of

$$\hat{x}_m(t) = x_m(t)[1 + 0.1\mathrm{Sin}(5.5\omega_n t)]$$

were produced which served as "noisy" observations of the system response. The search was then performed with the response $\hat{x}_m$, in order to determine how well the procedure would converge with scatter in the data. Damping coefficients of 19.29 lbs for the Coulomb term and 3.592 $\text{lb-sec}^2/\text{in}^2$ for the velocity squared term were obtained. Both of these quantities compare well with the assumed coefficients and both are within 3.5 per cent of the exact result.

General Dissipation Function

Returning to the case of noise free observations, an effort was made to describe the system in terms of other dissipation functions. The results of this analysis are given in Table 2. As can be seen, none of the other damping laws provided as close an agreement as did the true dissipation function.

Table 2. Results of the Optimum Search Procedure with the Mathematically Generated Response Data

Case No.	Damping Law	C_0	C_1	C_2	C_3	Criterion Function
Exact	C_0+C_2	20.0	-	3.5	-	-
1	C_0+C_2	19.54	-	3.599	-	1.930×10^{-5}
2	C_0	54.09	-	-	-	2.178×10^{-2}
3	C_1	-	18.44	-	-	2.546×10^{-3}
4	C_0+C_1	21.58	9.77	-	-	1.457×10^{-2}
5	C_1+C_2	-	13.65	1.581	-	6.410×10^{-3}
6	$C_0+C_1+C_2$	15.75	4.49	2.551	-	2.523×10^{-5}
7	$C_0+C_2+C_3$	16.18	-	4.041	0.0092	1.208×10^{-3}

An interesting point, which will be discussed in more detail in the next section, is apparent from these results. The correlation for all the various dissipation functions, except one, was much poorer than that assumed in Equation (5.1). This one exception, Case 6, which was a combination of Coulomb, viscous, and velocity squared damping, resulted in an agreement which was of the same order of magnitude as the true damping law. In this case, the answer was known before the fact, and this seemingly second valid dissipation function caused no problem. However, this same sort of phenomenon was observed with the physical model, and additional analysis and testing were required to determine which damping law was most valid.

Physically Measured Response Data

The transient response observations obtained with the experimental apparatus have also been analyzed with the developed computational algorithm. Two systems, one subsident and one oscillatory, were considered. In each case, the analysis was carried to a point which yielded an acceptable description of the system over a particular range of initial displacements.

Subsident System

Numerical Integration Criterion Function. Measured response, for three different initial conditions, was obtained for a system with the mass and spring rate as shown in Table 1. The search procedure, in conjunction with the Numerical Integration Criterion Function, was employed using the observations obtained for an initial displacement of 1.84 inches. As can be seen in Figure 27, neither viscous nor Coulomb damping alone allows adequate duplication of the motion of the mass. Various combinations of damping terms, Cases 3, 4, 5, and 7, Table 3,

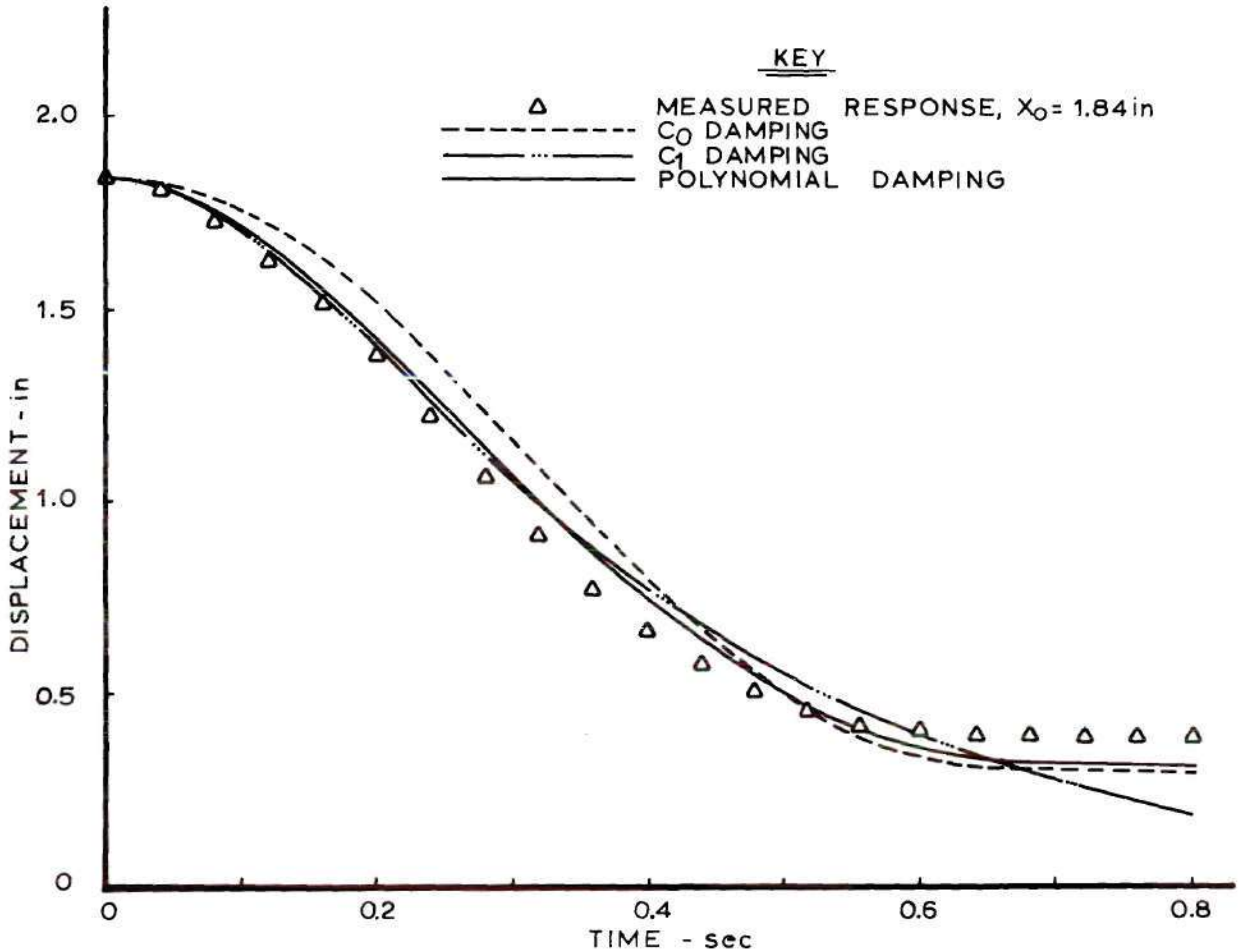


Figure 27. Displacement vs. Time—Subsident System

gave values for the criterion function which were of the same order of magnitude. The differences between the response predicted by these configurations were quite small and they are all represented as the solid line in Figure 27. At this point, none of these damping laws appeared to be significantly better than the next. This is the type of question which was indicated in the preceding section.

Table 3. Results of the Optimum Search Procedure with Physical Response Data--Subsident System

Case No.	Damping Law	C_0	C_1	C_2	C_3	C_4	Criterion Function
1	C_0	.05422	-	-	-	-	.2191
2	C_1	-	.02115	-	-	-	.1697
3	C_0+C_1	.03037	.00887	-	-	-	.0764
4	C_0+C_2	.03889	-	.00192	-	-	.0871
5	$C_0+C_1+C_2$	.03804	.00561	.00012	-	-	.0875
6	$C_0+C_1+C_2+C_3$	.04047	-.00311	.00026	.00090	-	.1200
7	$C_0+C_1+C_2+C_3+C_4$	.03343	.01157	.00247	.00004	.00011	.0818

NOTE: Initial Displacement = 1.84 in.

The fact that Case 6 includes the negative coefficient C_1, causes no concern at this point. It was assumed only that the total damping function dissipate energy. Nothing was assumed as to the nature of the individual terms.

In order to determine which damping law should be chosen, the measured response for the additional two initial displacements was used. The least squares criterion functions, shown in Table 4, were obtained for these remaining initial conditions. The criterion functions for each dissipation function were added together and these sums were compared with one another. The damping law resulting in the lowest sum was the final choice for the description of the system's unknown parameters. A combination of Coulomb and velocity squared damping gave the best comparison over the range of initial conditions considered. The final form for the system's equation of motion is expressed as

$$0.002137\ddot{x} + (0.03889 + 0.00192\dot{x}^2)\mathrm{Sgn}(\dot{x}) + 0.05042x = 0$$

A comparison of the above system description with the various sets of measured response is shown in Figure 28.

Table 4. Comparison of Criterion Functions for Various Initial Conditions--Subsident System

Initial Displacement	C_0+C_1	C_0+C_2	$C_0+C_1+C_2$	$C_0+C_1+C_2+C_3+C_4$
1.46 in	.0801	.0725	.2016	.1240
1.84 in	.0764	.0871	.0875	.0818
2.22 in	.1157	.0666	.2334	.0596
Sum	.2722	.2262	.5225	.2654

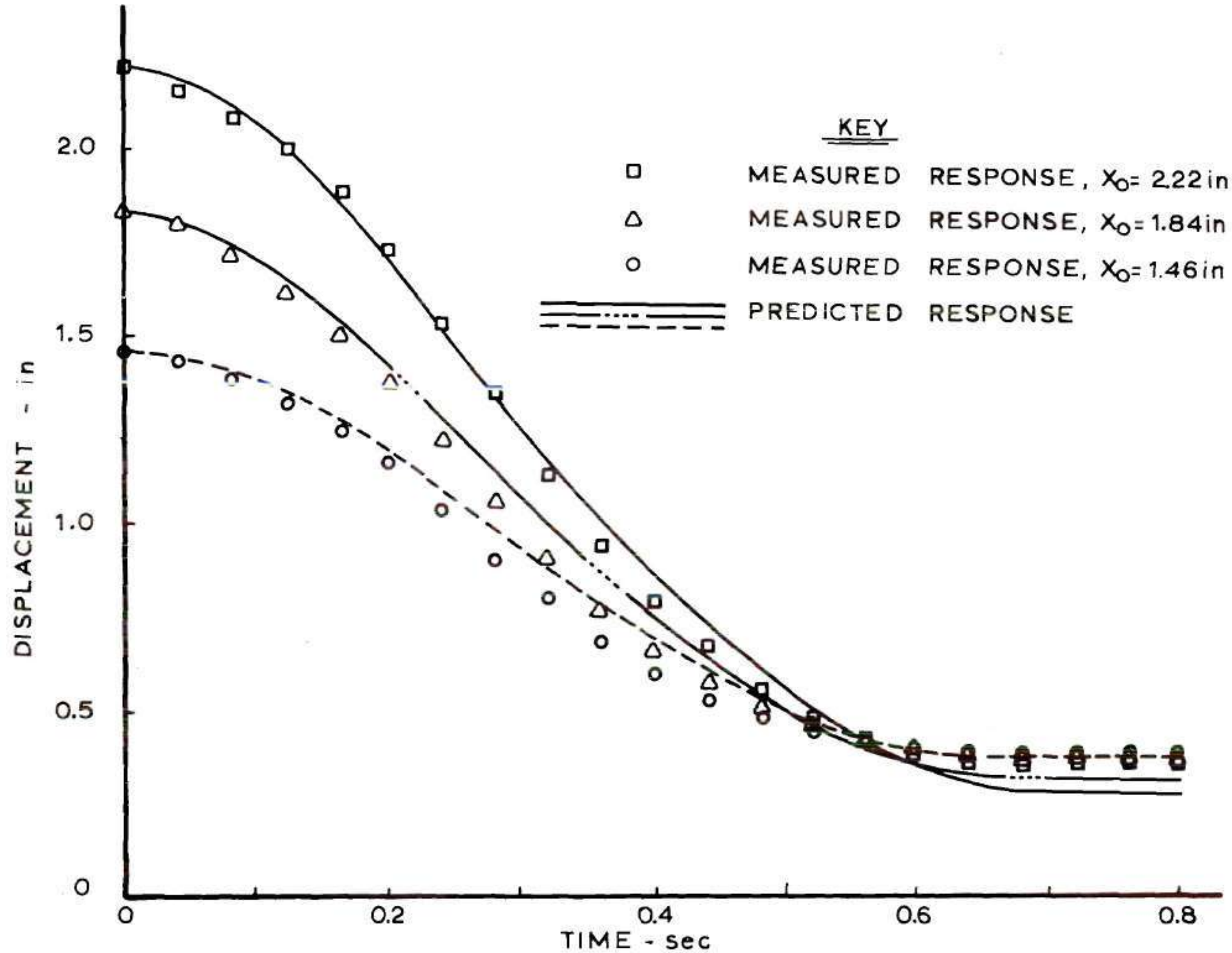


Figure 28. Displacement vs. Time--Subsident System

Approximate Solution Criterion Function. In addition, this set of system observations was evaluated in conjunction with the Approximate Solution Criterion Function. The Extended K-B Approximation with the viscous coefficient limited to less than critical was used. The analysis was performed for various combinations of Coulomb, viscous, and velocity squared terms.

Table 5. Results of Optimum Search Procedure with Physical Response Data--Subsident System--Approximate Solution Criterion Function

Case No.	Damping Law	C_0	C_1	C_2	Criterion Function	Corrected Criterion Function
1	C_0	.05419	-	-	.2274	-
2	C_1	-	.02076	-	.1729	-
3	C_0+C_1	.03117	.00858	-	.0779	-
4	C_0+C_2	.04523	-	.00112	.0279	.1019
5	$C_0+C_1+C_2$	.03279	.00641	.00095	.0084	.1262

NOTE: Initial Displacement = 1.84 in.

Table 5 lists the results from this application of the search procedure for an initial condition of 1.84 inches. The various combinations of viscous and Coulomb friction (Cases 1, 2, and 3) compare well with their respective cases in Table 3. Case 2, with viscous damping alone, does not compare as well because this coefficient was limited to less than critical in the approximation used. However, the configurations (Cases 4 and 5) with the nonlinear term C_2 present, did not yield good comparisons with the earlier results. The criterion functions shown in Table 5 were those obtained from the search procedure with the approximate representation of the response. The corrected criterion function was obtained by numerically integrating the equation of motion with the indicated damping constants and comparing this response with the measured response. Apparently the nonlinearities in this case were too strong for the approximation to adequately predict the system response. The computational time required in this instance was significantly less than with the numerical integration method. With the first three dissipation functions, this approach gave good results and this savings of computer time could be an important factor in these cases.

Oscillatory System

A second physical system with oscillatory motion has also been considered. The physical properties of this system are given in the second column of Table 1. Response information was obtained for this system with initial displacements of 0.46, 0.61, and 1.01 inches. All of the following discussions are for the evaluation of the criterion function with the numerical integration routine.

The search procedure was employed using the observations obtained for an initial displacement of 0.61 inches. The results of this initial search are shown in Figure 29 and Table 6. It is seen that viscous damping alone or several forms of a polynomial dissipation function all

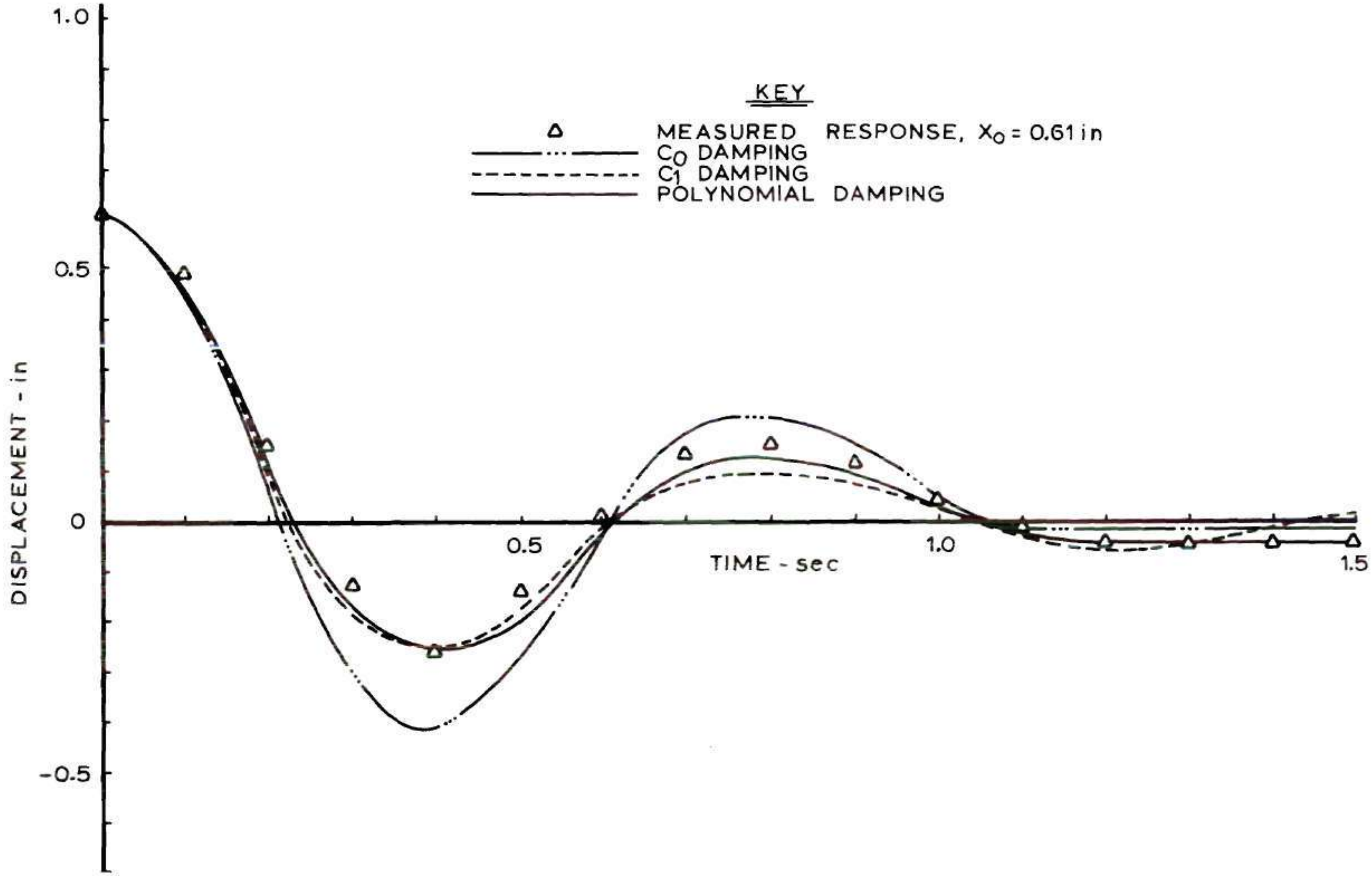


Figure 29. Displacement vs. Time--Oscillatory System

give a comparison within the same order of magnitude. Referring to Table 6, it is seen that Cases 2, 3, 4, 5, and 6 require additional consideration at this point. Cases 1 and 7 were neglected due to the large magnitudes of their criterion functions.

Table 6. Results of the Optimum Search Procedure with Physical Response Data--Oscillatory System

Case No.	Damping Law	C_0	C_1	C_2	C_3	C_4	Criterion Function
1	C_0	.009503	-	-	-	-	.08836
2	C_1	-	.006207	-	-	-	.02060
3	C_0+C_1	.001444	.005220	-	-	-	.01967
4	C_0+C_2	.003326	-	.001511	-	-	.01166
5	$C_0+C_1+C_2$	.002630	.002100	.001023	-	-	.01375
6	$C_0+C_1+C_2+C_3$	.005476	-.003887	.000770	.000973	-	.01144
7	$C_0+C_1+C_2+C_3+C_4$	.006106	-.007466	.000659	.000960	.000981	.08166

NOTE: Initial Displacement = 0.61 in.

The response of the system for the other initial displacements was then predicted using the remaining five damping laws. Table 7 indicates the results of determining the least squares criterion function for these additional cases. From this table, it appears that a combination of Coulomb and viscous damping gave the best comparison over the range of initial conditions of interest. Therefore, the system's equation of motion is

$$0.0014\ddot{x} + 0.001444\ \mathrm{Sgn}(\dot{x}) + .00522\dot{x} + 0.09623x = 0$$

A comparison of the above system description with the measured response is given in Figure 30.

Table 7. Comparison of Criterion Functions for Various Initial Conditions--Oscillatory System

Initial Displacement	C_1	C_0+C_1	C_0+C_2	$C_0+C_1+C_2$	$C_0+C_1+C_2+C_3$
0.46	.02671	.02230	.02488	.01770	.02880
0.61	.02060	.01967	.01166	.01375	.01144
1.01	.05077	.02644	.10646	.09465	.45037
Sum	.09808	.06841	.14302	.12610	.49061

Summary

The developed computational technique provides a useful method for converting observations on a physical system into a description of its dissipation function. An approximate representation of the solution of the equation of motion required much less computer time, but was limited by the very nature of the approximations made. The numerical approach was more general in its application and its form did not depend on the magnitude of the viscous coefficient.

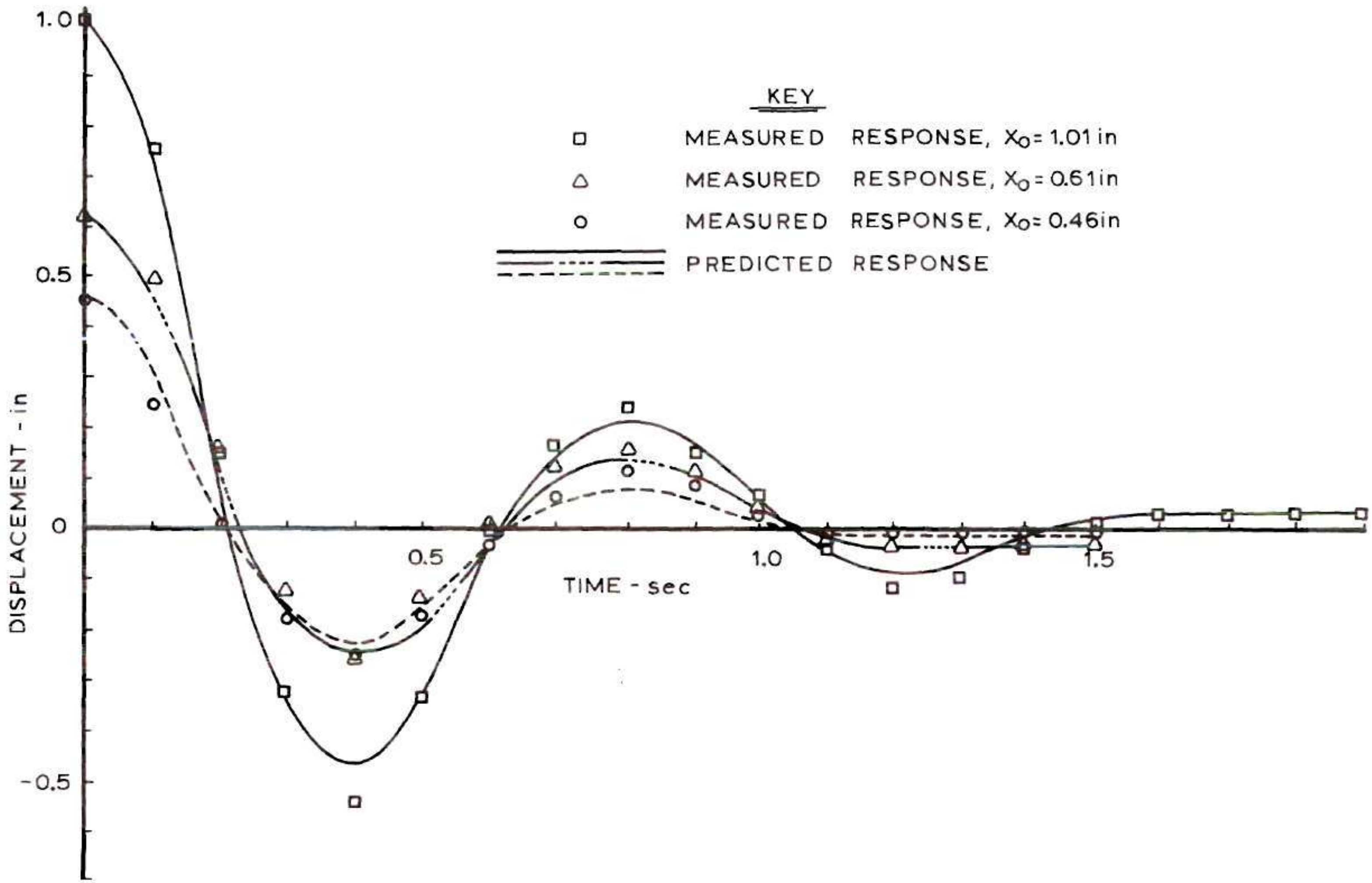


Figure 30. Displacement vs. Time--Oscillatory System

Often the best description of the dissipation function was not apparent with a single set of response data. After the form of the function has been reduced to two or three possible expressions, the mathematical solution was compared with the physical response for the other initial conditions. In this manner, the best representation of the damping law over the range of initial conditions was determined. If this analysis still does not indicate a particular choice, it would seem reasonable to pick the less complex dissipation function for the description of the system.

A discussion of a second approach to the searching procedure is presented in Appendix E. Here, the possibility of searching on all the transient data obtained and averaging the resulting system parameters is considered.

A characteristic of the search procedure which has not been discussed concerns the rate of convergence. When the search was located some distance away from the minimum point, the movement was quite rapid. However, as the extremum was approached, the speed of the process was reduced. Also, near the minimum point, the magnitude of the stepping variable λ became important. If this term was made too large, the search jumped back and forth across the location of the extremum.

As was shown with the mathematical response information, the presence of a small amount of noise in the measured signal did not seem to appreciably affect the final result.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER INVESTIGATION

Conclusions

Two techniques have been developed for obtaining an approximate solution for a dynamic system whose equation of motion contains a nonlinear dissipation function. The dissipation function that was considered consisted of a polynomial in the velocity. The first of these approximations was an extension of the Kryloff-Bogoliuboff technique and the second was a further application of a perturbation series method. These results are more general than previous efforts in the area since no limitations were placed on the magnitudes of the Coulomb and viscous damping coefficients.

Systems with both subsident and oscillatory motions were considered. The forms of the approximations were arranged according to the magnitude of the viscous damping coefficient. Of the two methods developed, the Extended K-B Approximation gave the best approximation when the linear term was less than critical, while the perturbation series approach was better in the overdamped case.

Secondly, a method has been formulated for converting observations on a nonlinear dynamic system into estimates of its unknown parameters. A computational algorithm employing an optimization technique was used to determine these unknown parameters. This procedure used a least squares approach to determine the measure of agreement between the mathematical model and the physical system. The response of the mathematical system was expressed both in terms of an approximate solution and a numerical integration technique. The numerical description of the response required greater computer time, but its application was not limited by the magnitude of any of the terms in the dissipation function. This procedure would be very helpful in extending the analysis of a physical system beyond the linear case.

Recommendations

With the viscous term less than critical, the derived Perturbation Series Approximation was quite limited in its area of application. The possibility of including more terms in the series approximation might counteract this limitation. Hopefully, this approximation could be extended to treat systems with larger nonlinearities.

During this investigation a question was raised concerning the identification process. Decisions on the possible forms of the dissipation function had to be made on a trial and error basis and each damping law had to be evaluated individually. Once a decision was made concerning the description of the system, there existed no assurance that there was not a more complex damping law which would provide better agreement with the measured response. It would be very helpful if some sort of analysis could be performed which would yield information on the form of the dissipation function without requiring a trial and error approach. The possibility of doing this in conjunction with the approximate solutions has been investigated, but no useful result was obtained.

The criterion function that was used in this study measured the agreement of the mathematical model in terms of the displacement of the system. The criterion function could be extended to include the effect of the system's velocity or acceleration. This new criterion function could employ some weighted measure including any combination of the displacement, velocity, and acceleration. The comparison between the physical system and the mathematical system should be made on the quantities which are important in terms of the over-all system performance.

As has been mentioned, the convergence of the computer routine was quite slow near the end of the identification process. The possibility of increasing this rate of convergence could be investigated. The magnitudes of the slopes of the criterion function response surface could be monitored in order to decide when the program logic should be transferred to some sort of refined descent method.

This entire investigation has been concerned with a dynamic system that was nonlinear in its dissipation function. There is no reason why the developed identification process could not be extended to a system which also contained an unknown nonlinear restoring force. Also, the nonlinear dissipation function that has been considered was a polynomial in the system's velocity. This analysis could be extended to include a dissipation function defined as

$$F(\dot{x}) = C_N \dot{x}^N$$

where both the coefficient C_N and the factor N would be the unknown parameters to be determined.

APPENDIX

APPENDIX A

MOTION DAMPED BY A COMBINATION OF COULOMB AND VISCOUS DAMPING

When a dynamic system includes a combination of Coulomb and viscous damping, it is possible for the motion to be subsident in nature even though the linear damping term is less than critical. This property is of importance when attempting to evaluate the unknown damping constants of a system when there is Coulomb friction present.

Mathematical Derivation

Consider, for example, a system governed by the following differential equation

$$\ddot{x} + \frac{C_o}{m} \mathrm{Sgn}(\dot{x}) + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = 0 \tag{A.1}$$

where

$$\zeta < 1.0$$

The solution to the above equation is

$$x(t) = e^{-\zeta\omega_n t}[X_o + \frac{C_o}{K} \mathrm{Sgn}(\dot{x})] \cdot [\frac{\zeta}{\sqrt{1-\zeta^2}} \mathrm{Sin}\omega_d t + \mathrm{Cos}\omega_d t] - \frac{C_o}{K} \mathrm{Sgn}(\dot{x}) \tag{A.2}$$

and the velocity expression is

$$\dot{x}(t) = -\frac{\omega_n}{\sqrt{1-\zeta^2}}[X_o + \frac{C_o}{K}\operatorname{Sgn}(\dot{x})]e^{-\zeta\omega_n t}\operatorname{Sin}\omega_d t$$

Here the system's initial conditions were assumed to be

$$x(o) = X_o \quad \text{and} \quad \dot{x}(o) = 0$$

It is now of interest to determine the condition which is required for subsident motion. In other words, this is the condition that is required for the system's displacement and velocity to both be zero at the same instant of time. This is expressed as

$$x(t^*) = \dot{x}(t^*) = 0$$

From the velocity expression, the following is obtained

$$\operatorname{Sin}\omega_d t^* = 0$$

or

$$\omega_d t^* = n\pi \quad \text{for } n = 0,1,2,3,\cdots$$

The case of n = o is trivial, and therefore the time at which the system returns to its equilibrium position is

$$t^* = \frac{\pi}{\omega_d} \tag{A.3}$$

Combining the displacement expression given in Equation (A.2) with the above time relationship yields

$$-[X_o + \frac{C_o}{K} Sgn(\dot{x})]e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - \frac{C_o}{K} Sgn(\dot{x}) = 0$$

With the assumed initial conditions, the velocity will always be negative until the system comes to rest. Therefore, the critical Coulomb damping coefficient, which will assure subsident motion, is given by

$$C_o = \frac{KX_o}{1 + e^{\zeta\pi/\sqrt{1-\zeta^2}}}$$

The effect of the addition of this critical amount of Coulomb damping to an underdamped system is shown in Figure 31.

If the amount of Coulomb damping present is more than the critical value, the system will come to rest before it reaches its undamped equilibrium position. This rest position is designated by x_s and at time t^* the following will hold

$$x(t^*) = x_s \quad \text{and} \quad \dot{x}(t^*) = 0$$

From the velocity expression, the value for t^* is the same as that given in Equation (A.3), but the displacement expression becomes

$$x_s = -[X_o + \frac{C_o}{K} Sgn(\dot{x})]e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - \frac{C_o}{K} Sgn(\dot{x})$$

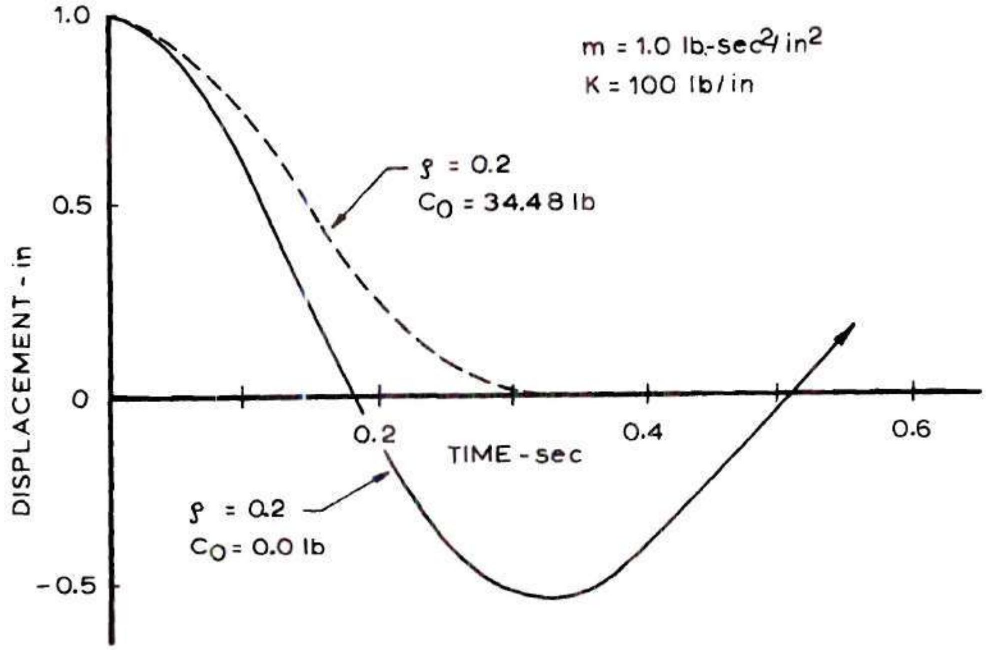


Figure 31. Motion Damped by a Combination of Coulomb and Viscous Damping

Therefore, in this case, the Coulomb damping coefficient is given by

$$C_o = K(x_s + X_o e^{-\zeta\pi/\sqrt{1-\zeta^2}})/(1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}})$$

Also note that Equation (A.2) reduces to

$$x(t) = [X_o + \frac{C_o}{K}\,\mathrm{Sgn}(\dot{x})]\mathrm{Cos}\omega_n t - \frac{C_o}{K}\,\mathrm{Sgn}(\dot{x})$$

if there is no viscous damping term present. In this case, the value of Coulomb damping required for subsident motion is given by

$$C_o = \frac{1}{2} KX_o$$

When the system comes to rest before it reaches its equilibrium position, the Coulomb damping coefficient is

$$C_o = \frac{1}{2} K(X_o+x_s)$$

APPENDIX B

DETAILS OF THE PERTURBATION SERIES APPROXIMATIONS

The generating solution or zero order term $\phi_o(t)$ can easily be obtained for all the various damping configurations which have been considered. The purpose of this appendix is to detail the steps required in obtaining the first order correction term $\phi_1(t)$. This term is the solution to the differential equation given earlier as Equation (2.19). This relationship is repeated below.

$$\ddot{\phi}_1 + 2\zeta\omega_n\dot{\phi}_1 + \omega_n^2\phi_1 + \gamma_2 \mathrm{Sgn}(\dot{x})\dot{\phi}_o^2 + \gamma_3\dot{\phi}_o^3 + \cdots = 0 \qquad (2.19)$$

Overcritical Viscous Damping

From Equation (2.18), the velocity associated with the generating solution is given by

$$\dot{\phi}_o = \frac{\alpha\beta}{\alpha - \beta} X^*(e^{\beta t} - e^{\alpha t})$$

The general expression for the nth power of this velocity is

$$\dot{\phi}_o^n = \sum_{i,j=0,1,2\cdots}^{n} (-1)^j \left(\frac{\alpha\beta X^*}{\alpha - \beta}\right)^{i+j} D_{ij}\, e^{(i\beta + j\alpha)t}$$

where the constants D_{ij} are given in Table 8 and the relationship

Table 8. Constants D_{ij} for the Perturbation Series Approximation--Overcritical Viscous Damping

i/j	0	1	2	3	4
0	-	-	1	1	1
1	-	2	3	4	5
2	1	3	6	10	15
3	1	4	10	20	35
4	1	5	15	35	80

$2 \leq i + j \leq n$ must always be satisfied. Now expressing d_{ij} as

$$d_{ij} = \begin{cases} D_{ij}\gamma_{i+j}\text{Sgn}(\dot{x}) & \text{for } i+j = 2,4,6,\cdots \\ D_{ij}\gamma_{i+j} & \text{for } i+j = 3,5,\cdots \end{cases}$$

the equation governing $\phi_1(t)$ may now be expressed as

$$\ddot{\phi}_1 + 2\zeta\omega_n\dot{\phi}_1 + \omega_n^2\phi_1 + \sum_{i,j=0,1,2,\cdots}^{n} (-1)^j \left(\frac{\alpha\beta \ X^*}{\alpha - \beta}\right)^{i+j} d_{ij} \ e^{(i\beta+j\alpha)t} = 0 \tag{B.1}$$

The complementary solution to the above is

$$\phi_1 = A_1e^{\alpha t} + B_1e^{\beta t}$$

The following is now assumed to be the form of the particular integral of Equation (B.1)

$$\phi_1 = \sum_{i,j=0,1,2,\cdots}^{n} C_{ij}\, e^{(j\alpha+i\beta)t}$$

Therefore

$$\dot{\phi}_1 = \sum_{i,j=0,1,2,\cdots}^{n} (j\alpha+i\beta)C_{ij}\, e^{(j\alpha+i\beta)t}$$

and

$$\ddot{\phi}_1 = \sum_{i,j=0,1,2} (j\alpha+i\beta)^2 C_{ij}\, e^{(j\alpha+i\beta)t}$$

Inserting these in Equation (B.1) and solving for C_{ij} results in

$$C_{ij} = \frac{(-1)^{i+j}\left(\frac{\alpha\beta\ X^*}{\alpha - \beta}\right)^{i+j}}{[(j\alpha+i\beta)^2 + 2\zeta\omega_n(j\alpha+i\beta) + \omega_n^2]}$$

where

$$i,j = 0,1,2,\cdots \qquad \text{and} \qquad 2 \le i + j \le n$$

The total solution for $\phi_1(t)$ becomes

$$\phi_1 = A_1 e^{\alpha t} + B_1 e^{\beta t} + \sum_{i,j=0,1,2,\cdots}^{n} C_{ij}\, e^{(j\alpha+i\beta)t}$$

subject to the initial conditions of Equation (2.16). With these initial conditions the following relations hold.

$$0 = A_1 + B_1 + \sum_{i,j=0,1,2,3}^{n} C_{ij}$$

$$0 = \alpha A_1 + \beta B_1 + \sum_{i,j=0,1,2,\cdots}^{n} C_{ij}(j\alpha+i\beta)$$

The above can be solved for the unknown coefficients A_1 and B_1 and the final expression for $\phi_1(t)$, as given in Equation (2.20), can be obtained.

Undercritical Viscous Damping

The velocity associated with the generating solution is given as

$$\dot{\phi}_o = -\frac{\omega_n X^*}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \operatorname{Sin}\omega_d t$$

Therefore, the successive powers of the velocity become

$$\dot{\phi}_o^2 = \frac{1}{2} \left(\frac{\omega_n X^*}{\sqrt{1-\zeta^2}}\right)^2 e^{-2\zeta\omega_n t} (1-\operatorname{Cos}2\omega_d t) \qquad (B.2)$$

$$\dot{\phi}_o^3 = \frac{1}{4} \left(\frac{\omega_n X^*}{\sqrt{1-\zeta^2}}\right)^3 e^{-3\zeta\omega_n t} (3\operatorname{Sin}\omega_d t-\operatorname{Sin}3\omega_d t)$$

$$\dot{\phi}_o^4 = \frac{1}{8} \left(\frac{\omega_n X^*}{\sqrt{1-\zeta^2}}\right)^4 e^{-4\zeta\omega_n t} (3-4\operatorname{Cos}2\omega_d t+\operatorname{Cos}4\omega_d t)$$

At this point, the method of Laplace transformations was used to solve Equation (2.19) for $\phi_1(t)$. Defining y(s) as the transform of

$\phi_1(t)$, the following notation has been used

$$y(s) = L\{\phi_1(t)\}$$

Transforming the governing differential equation, Equation (2.19), and including the initial conditions of Equation (2.16) gives

$$s^2 y(s) + 2\zeta\omega_n s y(s) + \omega_n^2 y(s) + \gamma_2 \mathrm{Sgn}(\dot{x}) L\{\dot{\phi}_o^2\} +$$

$$+ \gamma_3 L\{\dot{\phi}_o^3\} + \cdots = 0$$

From this, the transform of the response becomes

$$y(s) = \left(\frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \left(\gamma_2 \mathrm{Sgn}(\dot{x}) L\{\phi_o^2\} + \gamma_3 L\{\phi_o^3\} + \cdots\right)$$

Employing the relationships of Equations (B.2) and then taking the inverse transformation of the above yields the expression for the first order correction term. This response function is presented as Equation (2.23).

APPENDIX C

CONSIDERATION OF SECULAR TERMS IN THE PERTURBATION SERIES APPROXIMATION

As was indicated in Chapter II, under certain conditions, the techniques employed in obtaining the Perturbation Series Approximation may result in secular terms. These conditions correspond to an equation of motion which lacks a viscous damping term. In this case, secular terms will result with the presence of damping terms proportional to odd powers of the velocity. This complication did not arise in the approximations developed in Chapter II, since these were formulated with a C_1 coefficient present. This coefficient was then allowed to go to zero if that was the case being considered.

An example of this complication and a technique for treating it are discussed here. Consider a system possessing a dissipation function consisting of only a velocity cubed term. The system's equation of motion is

$$\ddot{x} + \varepsilon\dot{x}^3 + \omega_n x = 0 \tag{C.1}$$

where

$$\varepsilon = \frac{C_3}{m}$$

and with the initial conditions

$$x(o) = X_o$$

and

$$\dot{x}(o) = 0$$

Substituting the assumed power series form for $x(t)$, Equation (2.14), into the above differential equation, and considering one correction term, results in

$$\varepsilon^0[\ddot{\phi}_o + \omega_n^2 \phi_o] + \varepsilon^1[\ddot{\phi}_1 + \omega_n^2 \phi_1 + \dot{\phi}_o^3] = 0$$

with the initial conditions

$$\phi_o(0) = X_o$$

and

$$\dot{\phi}_o(0) = \phi_1(0) = \dot{\phi}_1(0) = 0$$

Therefore, the solution for the zero order term is

$$\phi_o(t) = X_o \operatorname{Cos}\omega_n t$$

and

$$\dot{\phi}_o(t) = -\omega_n X_o \operatorname{Sin}\omega_n t$$

The equation for the first order correction term becomes

$$\ddot{\phi}_1 + \omega_n^2\phi_1 - \omega_n^3 X_o^3 \operatorname{Sin}^3\omega_n t = 0$$

The solution for this differential equation is

$$\phi_1(t) = \frac{9}{32}\,\omega_n X_o^3 \operatorname{Sin}\omega_n t - \frac{1}{32}\,\omega_n X_o^3 \operatorname{Sin}3\omega_n t - \frac{3}{8}\,\omega_n^2 X_o^3 t \operatorname{Cos}\omega_n t \qquad \text{(C.2)}$$

The last term in this expression is the troublesome secular term. It is of interest to note that by allowing only C_3 to appear in Equation (2.23), the relationship given in Equation (C.2) is obtained minus the secular term.

Instead of the series expansion employed in Equation (2.14), the following form is assumed for the representation of x(t).

$$x(t) = \varepsilon^0\phi_o(t_0,t_1) + \varepsilon^1\phi_1(t_0,t_1) \qquad \text{(C.3)}$$

where

$$t_n = \varepsilon^n t$$

Therefore

$$\frac{d}{dt} = \varepsilon^0 \frac{\partial}{\partial t_o} + \varepsilon^1 \frac{\partial}{\partial t_1}$$

As was done in the Perturbation Series Approximation, the above expressions are introduced into the equation of motion, the coefficients of like powers of ε are set equal to zero, and the resulting equations are solved. These solutions contain arbitrary functions of t_1 which are determined by requiring that $x(t)$ remain bounded for all time. The basis for this technique of eliminating secular terms was developed by Nayfeh (45).

Returning to the question of the system governed by Equation (C.1), the following results due to the assumptions of Equation (C.3)

$$\dot{x} = \frac{\partial \phi_0}{\partial t_0} + \varepsilon\left(\frac{\partial \phi_0}{\partial t_1} + \frac{\partial \phi_1}{\partial t_0}\right)$$

$$\ddot{x} = \frac{\partial^2 \phi_0}{\partial t_0^{\,2}} + \varepsilon\left(\frac{\partial^2 \phi_1}{\partial t_0^{\,2}} + 2\,\frac{\partial^2 \phi_0}{\partial t_0 \partial t_1}\right)$$

Inserting these expressions into the equation of motion gives

$$\varepsilon^0\left[\frac{\partial^2 \phi_0}{\partial t_0^{\,2}} + \omega_n^{\,2}\phi_o\right] + \varepsilon^1\left[\frac{\partial^2 \phi_1}{\partial t_0^{\,2}} + \omega_n^2\phi_1 + 2\frac{\partial^2 \phi_o}{\partial t_0 \partial t_1} \quad \left(\frac{\partial \phi_o}{\partial t_0}\right)^3\right] = 0$$

The solution to the first equation is

$$\phi_o(t_0,t_1) = A(t_1)\mathrm{Sin}\omega_n t_0 + B(t_1)\mathrm{Cos}\omega_n t_0$$

In order to satisfy the initial conditions of the system, the following are obtained

$$A(o) = 0 \quad \text{and} \quad B(o) = X_0$$

Therefore

$$\left(\frac{\partial \phi_0}{\partial t_0}\right)^3 = -\omega_n^{\,3}B^3(t_1)\mathrm{Sin}^3\omega_n t_0$$

$$= -\frac{1}{4}\,\omega_n^{\,3}B^3(t_1)(3\mathrm{Sin}\omega_n t_0-\mathrm{Sin}3\omega_n t_0)$$

and the second equation becomes

$$\frac{\partial^2\phi_1}{\partial t_0^{\,2}} + \omega_n^{\,2}\phi_1 - 2\frac{\partial}{\partial t_1}\,(\omega_n B(t_1)\mathrm{Sin}\omega_n t_0) -$$

$$-\frac{1}{4}\,\omega_n^{\,3}B^3(t_1)(3\mathrm{Sin}\omega_n t_0-\mathrm{Sin}3\omega_n t_0) = 0$$

with the initial conditions

$$\phi_1(0,0) = \frac{\partial}{\partial t_0}\ \phi_1(0,0) = 0$$

In order that x(t) be bounded, the secular producing terms must be eliminated from the above expression. Therefore

$$-2\omega_n\mathrm{Sin}\omega_n t_0\ \frac{dB(t_1)}{dt_1} - \frac{3}{4}\,\omega_n^{\,3}\mathrm{Sin}\omega_n t_0 B^3(t_1) = 0$$

where

$$B(0) = X_0$$

This yields the following relationship for the unknown function $B(t_1)$

$$B(t_1)^2 = \frac{X_0^{\,2}}{1 + \frac{3}{4}\,\omega_n^{\,2} X_0^{\,2} t_1} \tag{C.4}$$

The differential equation for the first order correction term is

$$\frac{\partial^2 \phi_1}{\partial t_0^{\,2}} + \omega_n^{\,2} \phi_1 = -\frac{1}{4}\,\omega_n^{\,3} \left[\frac{X_0^{\,2}}{1 + \frac{3}{4}\ {}_n^{\,2} X_0^{\,2} t_1}\right]^{3/2} \operatorname{Sin}3\omega_n t_0$$

Consistent with the initial conditions imposed on $\phi_1(t_0,t_1)$, the following is obtained

$$\phi_1(t_0,t_1) = -\frac{3\omega_n}{32} \left[\frac{X_0^{\,2}}{1 + \frac{3}{4}\,\omega_n^{\,2} X_0^{\,2} t_1}\right]^{3/2} [\operatorname{Sin}\omega_n t_0 - \frac{1}{3}\operatorname{Sin}3\omega_n t_0]$$

Therefore, the modified perturbation series approximation for the system response including one correction term becomes

$$x(t) = \left[\frac{X_0^{\,2}}{1 + \frac{3}{4}\frac{C_3}{m}\,\omega_n^{\,2} X_0^{\,2} t}\right]^{1/2} \operatorname{Cos}\omega_n t \; -$$

$$-\frac{3}{32}\frac{C_3}{m}\,\omega_n \left[\frac{X_0^{\,2}}{1 + \frac{3}{4}\frac{C_3}{m}\,\omega_n^{\,2} X_0^{\,2} t}\right]^{3/2} (\operatorname{Sin}\omega_n t - \frac{1}{3}\operatorname{Sin}3\omega_n t)$$

It is of interest to note that the first term in the above expression agrees with the result that is obtained when the classic Kryloff-Bogoliuboff approximation method is used.

APPENDIX D

REPRESENTATIVE COMPUTER PROGRAMS

The following presents a listing of the representative computer programs used during this investigation. These programs were for the Univac 1108 digital computer and are written in ALGOL language.

The first one given is the optimization procedure referred to as the Multidimensional Search Routine. This is followed by the two procedures which were used to evaluate the criterion function. The first of these is the Approximate Solution Criterion Function, followed by the Numerical Integration Criterion Function.

MULTIDIMENSIONAL SEARCH ROUTINE

```
          BEGIN
COMMENT   *  *  *  *  *  *  *
          PROGRAM 211 ( 5-30-68).
          MULTIDIMENSIONAL SEARCH ROUTINE.
          CONVERGENCE BY THE GOLDEN-SECTION.
          UNIVAC 1108 VERSION ( 4-13-68).
          R.M. LAURENSON.
          M.E. DEPT. - EXT. 5153.
                NOTES -
                  1. WHEN USING THE PROCEDURE APPROX, (A) MULT = 1
                     AND (B) N,N1,AND KODE MUST BE SET.
                  2. GOOD RULE OF THUMB' DT*R1 LSS 2 FOR
                     OVERDAMPED CASE WHEN USING 'INTEGRATE'.
                  3. THE FOLLOWING IS THE ORDER OF THE INPUT DATA'
                     (1) M,K (2) DT,HALT,MULT,Q (3) CODE,LAMBDA,
                     RFACTOR,IFACTOR,FACTOR,SFACTOR(*) (4) C(*),
                     VARY(*) (5) XM(*).                          $
REAL            WN2,WN,DT,M,K,ZETA,L,LS,LLS,RATIO,RFACTOR,DL,LAMBDAS,
                LAMBDASAVE,SN,SGN,COUNT,ZETAO,LAMBDA,IFACTOR,HOLD,FACTOR
                ,CFSAVE,CFHOLD                                   $
INTEGER         HALT,SCRIBE,MULT,I,J,N,S,T,CODE,REDUCE,NN,Q,CON,NUMB,RX$
INTEGER         NUMB1                                            $
REAL ARRAY      R,C,SFACTOR,CB,CFS,SLOPE,CHOLD,CSAVE(0'4),RF(0'4,0'4),
                CS(0'4,0'3),CF(0'3)                              $
INTEGER ARRAY   VARY(0'4)                                        $
FORMAT          FORM1(X6,'PROGRAM 211 ( 5-30-68)',A0.1,X6,
                    'MULTIDIMENSIONAL SEARCH ROUTINE',A0.1,X6,
                    'UNIVAC 1108 VERSION ( 4-13-68)',A0.3),
                FORM2(X6,'PHYSICAL DATA'',A0.1,X14,'M',X15,'K',X14,'WN',
                    X13,'ZETA',A0.1,X4,4(X4,R12.6),A0.2),
                FORM3(X6,'MEASURED RESPONSE'',A0.1,(X2,5(X4,D12.6),A0.1,
                    ),A0.1),
                FORM4(X6,'LOGIC DATA'',A1.1,X35,'SCALE FACTOR'',A0.1,
                    X2,5(X4,R12.6),A0.2,X12,'LAMBDA',X10,'RFACTOR',
                    X5,'INTERVAL FACTOR',X5,'FACTOR',X12,'DT',A0.1,
                    X2,3(X4,D12.5),X7,R12.6,X1,D12.5,A0.2,X11,'CODE',
                    X3,'MULT',X3,'HALT',
                    X10,'VARY',A0.1,X13,I1,X4,I3,X5,I2,X5,5(X2,I1),
                    A0.4,X29,'DAMPING COEFFICIENTS'',X42,'CRITERION',
                    A0.1,X12,'C0',X14,'C1',X14,'C2',X14,'C3',X14,'C4',
                    X14,'FUNCTION'',A0.1),
                FORM5(X2,5(X4,R12.6),X8,R12.6,A0.1),
                FORM6(X6,'WILL REVERSE DIRECTION ALONG THE GRADIENT'',
                    A1.1),
                FORM7(X6,'THE FOLLOWING IS THE RESULT OF THE ',
                    'GOLDEN-SECTION'',A1.1),
                FORM8(X6,'RETURN LOGIC TO MOVEMENT DOWN ',
                    'GRADIENT'',A1.1),
                FORM9(X6,'THE FOLLOWING IS THE CALCULATED RESPONSE',
```

```
                 ' OF THE SYSTEM'',A0.2),
            FORM10(X6,'FINAL CRITERION FUNCTION = ',R12.6,A0.2,X6,
                  'FINAL DAMPING COEFFICIENTS'',A0.1,X10,'C0 = ',
                  R12.6,A0.1,X10,'C1 = ',R12.6,A0.1,X10,'C2 = ',
                  R12.6,A0.1,X10,'C3 = ',R12.6,A0.1,X10,'C4 = ',
                  R12.6,A0.2,X10,'ZETA = ',R12.6,A0.1,X10,
                  'ZETA0 = ',R12.6,A0.1),
            FORM11(E0,A1),
            FORM12(X6,'THE FOLLOWING ARE THE SLOPES'',A1.1,
                  X2,5(X4,R12.6),A0.2),
            FORM13(X6,'SEARCH FAILS TO CONVERGE - WILL TABULATE ',
                  'THE BEST RESULTS'',A0.2),
            FORM14(X6,'CAN NOT IMPROVE ON CFSAVE - WILL ',
                  'TABULATE THE BEST RESULTS'',A0.2)                 $
COMMENT     *  *  *  *  *  *  *
            PLACE PROCEDURE 'ERROR' HERE'
            *  *  *  *  *  *  *                                         $
```

THE APPROPRIATE PROCEDURE FOR EVALUATING
THE CRITERION FUNCTION IS PLACED HERE.

```
COMMENT     *  *  *  *  *  *  *
            BEGINNING OF THE READING OF THE INPUT DATA AND THE
            CALCULATION OF THE PHYSICAL CONSTANTS.
            *  *  *  *  *  *  *                                         $
MAIN'
               READ(CARDS,M,K,DT,HALT,MULT,Q,CODE,LAMBDA,RFACTOR,
                  IFACTOR,FACTOR,SFACTOR,C,VARY)                     $
            BEGIN
REAL ARRAY     XM(0'Q)                                                  $
               READ(CARDS,XM)                                           $
               WRITE(FORM1)                                             $
               WN2 = K/M                                                $
               WN = SQRT(WN2)                                           $
               ZETA = C(1)/(2.0*WN*M)                                   $
               WRITE(FORM2,M,K,WN,ZETA)                                 $
               WRITE(FORM3,XM)                                          $
               WRITE(FORM4,SFACTOR,LAMBDA,RFACTOR,IFACTOR,FACTOR,DT,
                  CODE,MULT,HALT,VARY)                                  $
COMMENT     *  *  *  *  *  *  *
            END OF THE READING OF THE INPUT DATA AND THE CALCULATION
            OF THE PHYSICAL CONSTANTS.
            BEGINNING OF THE SEARCH ROUTINE.
            *  *  *  *  *  *  *                                         $
               SCRIBE = 0                                               $
               L = 1.0                                                  $
               LAMBDASAVE = LAMBDA                                      $
               CFHOLD = -100.0                                          $
               CFSAVE = -100.0                                          $
               CON = 0                                                  $
```

```
          NUMB = 0                                                    $ 80B
          NUMB1 = 0                                                   $ 80BA
          RX = 0                                                      $ 80C
     FOR  I = 0 STEP 1 UNTIL 4 DO                                       81
     BEGIN                                                              82
          R(I) = C(I)/M                                               $ 83
          RF(I,I) = RFACTOR                                           $ 84
          CS(I,0) = SFACTOR(I)*C(I)                                   $ 85
          SLOPE(I) = 0.0                                              $ 85A
     END                                                              $ 86
          CF(0) = ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C(1),M)          $ 87
          CS(1,0) = SFACTOR(1)*C(1)                                   $ 87A
POINT1'                                                                 88
          WRITE(FORM5,C,CF(0))                                        $ 89
          LAMBDA = 0.999999*LAMBDASAVE                                $ 90
          LAMBDAS = LAMBDA                                            $ 91
          SN = -1.0                                                   $ 92
          REDUCE = 1                                                  $ 93
          NN = 0                                                      $ 94
          COUNT = 1.0                                                 $ 95
          IF RX EQL 1 THEN GO TO POINT2                               $ 95A
COMMENT  *  *  *  *  *  *  *                                            96
     THIS IS THE BEGINNING OF THE SCHEME TO DETERMINE THE SLOPES.       97
     *  *  *  *  *  *  *                                              $ 98
POINT1A'                                                                98A
     FOR  N = 0 STEP 1 UNTIL 4 DO                                       99
     BEGIN                                                             100
          IF VARY(N) EQL 1 THEN                                        101
     BEGIN                                                             102
     FOR  I = 0 STEP 1 UNTIL 4 DO                                      103
     BEGIN                                                             104
          CB(I) = CS(I,0)+RF(I,N)*CS(I,0)                             $105
          R(I) = CB(I)/(M*SFACTOR(I))                                 $106
     END                                                              $107
          CFS(N) = ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C(1),M)         $108
          SLOPE(N) = (CFS(N)-CF(0))/(CB(N)-CS(N,0))                   $109
     END                                                              $110
     END                                                              $111
          WRITE(FORM12,SLOPE)                                         $111A
          IF CON EQL 1 THEN GO TO POINT6A                             $111B
COMMENT  *  *  *  *  *  *  *                                           112
     END OF SCHEME FOR DETERMINING SLOPES.                             113
     BEGINNING OF MOVEMENT ALONG GRADIENT.                             114
     *  *  *  *  *  *  *                                              $115
POINT2'                                                                116
          SGN = SIGN(SN)                                              $117
     FOR  I = 0 STEP 1 UNTIL 4 DO                                      118
     BEGIN                                                             119
          CS(I,1) = CS(I,0)+SGN*LAMBDA*SLOPE(I)                       $120
          C(I) = CS(I,1)/SFACTOR(I)                                   $121
          R(I) = C(I)/M                                               $122
     END                                                              $123
          CF(1) = ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C(1),M)          $124
          CS(1,1) = SFACTOR(1)*C(1)                                   $124A
          WRITE(FORM5,C,CF(1))                                        $125
          IF CF(1) LSS CF(0) THEN                                      126
```

```
        BEGIN
              COUNT = COUNT+1.0
              LAMBDA = LAMBDAS*COUNT
              CF(0) = CF(1)
        FOR   I = 0 STEP 1 UNTIL 4 DO CS(I,0)= CS(I,1)
              GO TO POINT2
        END
              IF ABS((CF(0)-CF(1))/CF(0)) LSS RFACTOR THEN
              GO TO POINT3
              WRITE(FORM6)
              IF REDUCE NEQ 1 THEN
        BEGIN
              LAMBDA = LAMBDA/COUNT
              GO TO PT1
        END
              REDUCE = CODE
              LAMBDA = LAMBDA/(2.0*COUNT)
        PT1'
              COUNT = 1.0
              LAMBDAS = LAMBDA
              IF NN NEQ 0 THEN
        BEGIN
              NN = 0
              SN = -1.0
              GO TO POINT2
         END
              NN = 1
              SN = +1.0
              GO TO POINT2
COMMENT * * * * * * * *
        END OF MOVEMENT ALONG GRADIENT.
        BEGINNING OF GOLDEN-SECTION SEARCH.
        * * * * * * * *
POINT3'
              WRITE(FORM7)
              RATIO = -SGN/LAMBDA
              LS = 0.382*L
              DL = LS/RATIO
        FOR   I = 0 STEP 1 UNTIL 4 DO
        BEGIN
              CS(I,2) = CS(I,0)+SGN*DL*SLOPE(I)
              R(I) = CS(I,2)/(M*SFACTOR(I))
        END
              CF(2) = ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C(1),M)
              CS(1,2) = SFACTOR(1)*C(1)
              LLS = 0.618*L
              DL = LLS/RATIO
        FOR   I = 0 STEP 1 UNTIL 4 DO
        BEGIN
              CS(I,3) = CS(I,0)+SGN*DL*SLOPE(I)
              R(I) = CS(I,3)/(M*SFACTOR(I))
        END
              CF(3) = ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C(1),M)
              CS(1,3) = SFACTOR(1)*C(1)
POINT4'
              L = LLS
```

```
          LLS = LS
          LS = 0.382*L
          DL = LS/RATIO
          IF CF(3) GTR CF(2) THEN GO TO POINT5
          CF(0) = CF(2)
          CF(2) = CF(3)
      FOR  I = 0 STEP 1 UNTIL 4 DO '
      BEGIN
          CS(I,0) = CS(I,2)
          CS(I,2) = CS(I,3)
          CS(I,3) = CS(I,0)+SGN*DL*SLOPE(I)
          R(I) = CS(I,3)/(M*SFACTOR(I))
      END
          CF(3) = ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C(1),M)
          CS(1,3) = SFACTOR(1)*C(1)
          GO TO POINT6
POINT5'
          CF(1) = CF(3)
          CF(3) = CF(2)
      FOR  I = 0 STEP 1 UNTIL 4 DO
      BEGIN
          CS(I,1) = CS(I,3)
          CS(I,3) = CS(I,2)
          CS(I,2) = CS(I,0)+SGN*DL*SLOPE(I)
          R(I) = CS(I,2)/(M*SFACTOR(I))
      END
          CF(2) = ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C(1),M)
          CS(1,2) = SFACTOR(1)*C(1)
POINT6'
          IF L GTR IFACTOR THEN GO TO POINT4
COMMENT  * * * * * * *
      END OF THE GOLDEN-SECTION SEARCH.
      BEGINNING OF THE SORT ROUTINE.
      * * * * * * *
      FOR  S = 3 STEP -1 UNTIL 0 DO
      FOR  T = 0 STEP  1 UNTIL S DO
      BEGIN
          IF CF(S) GTR CF(T) THEN GO TO PT2
          HOLD = CF(S)
          CF(S) = CF(T)
          CF(T) = HOLD
      FOR  I = 0 STEP 1 UNTIL 4 DO
      BEGIN
          CHOLD(I) = CS(I,S)
          CS(I,S) = CS(I,T)
          CS(I,T) = CHOLD(I)
      END
      PT2'
      END
      FOR  I = 0 STEP 1 UNTIL 4 DO C(I) = CS(I,0)/SFACTOR(I)
          WRITE(FORM5,C,CF(0))
          CON = 1
          GO TO POINT1A
COMMENT  * * * * * * *
      END OF THE SORT ROUTINE.
      BEGINNING OF SCHEME FOR DETERMINING CONVERGENCE.
```

```
            *  *  *  *  *  *  *
POINT6A'
            FOR  I = 0 STEP 1 UNTIL 4 DO
            BEGIN
                 IF ABS(SLOPE(I)) GTR FACTOR THEN CON = 0
            END
                 IF CON EQL 1 THEN GO TO POINT7
                 IF ABS((CF(0)-CFSAVE)/CF(0)) LSS 1.09-5 THEN
                           NUMB = NUMB+1 ELSE NUMB = 0
                 IF NUMB GEQ 5 THEN
            BEGIN
                 WRITE(FORM11)
                 WRITE(FORM13)
                 GO TO POINT7A
            END
                 IF CF(0) LSS ABS(CFSAVE) THEN
            BEGIN
            FOR  I = 0 STEP 1 UNTIL 4 DO  CSAVE(I) = C(I)
                 CFSAVE = CF(0)
                 GO TO POINT6B
            END
                 IF ABS((CF(0)-CFHOLD)/CF(0)) LSS 1.09-5 THEN
                           NUMB1 = NUMB1+1 ELSE NUMB1 = 0
                 IF NUMB1 GEQ 5 THEN
            BEGIN
                 WRITE(FORM11)
                 WRITE(FORM14)
                 GO TO POINT7A
            END
            FOR  I = 0 STEP 1 UNTIL 4 DO C(I) = CSAVE(I)
                 CFHOLD = CF(0)
                 CF(0) = CFSAVE
POINT6B'
                 L = 1.0
                 RX = 1
                 WRITE(FORM8)
                 GO TO POINT1
COMMENT     *  *  *  *  *  *  *
            END OF THE SCHEME FOR DETERMINING CONVERGENCE.
            END OF SEARCH ROUTINE.
            TABULATION OF FINAL RESULTS.
            *  *  *  *  *  *  *
POINT7'
                 WRITE(FORM11)
POINT7A'
                 WRITE(FORM9)
                 SCRIBE = 1
            FOR  I = 0 STEP 1 UNTIL 4 DO
            BEGIN
                 IF CF(0) GEQ ABS(CFSAVE) THEN C(I) = CSAVE(I)
                 R(I) = C(I)/M
            END
                 CF(0) = ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C(1),M)
                 WRITE(FORM11)
                 ZETA = C(1)/(2.0*WN*M)
                 ZETAO = (2.0*C(0))/(WN2*M*XM(0))
```

```
      WRITE(FORM10,CF(0),C,ZETA,ZETAO)                             $262
END                                                                $263
      WRITE(FORM11)    ,                                           $263A
      GO TO MAIN                                                   $264
END                                                                $265
```

APPROXIMATE SOLUTION CRITERION FUNCTION

```
COMMENT    *  *  *  *  *  *  *
           APPROXIMATE SOLUTION CRITERION FUNCTION.
           THE FOLLOWING PROCEDURE EVALUATES THE LEAST SQUARES
           CRITERION FUNCTION.
           UNIVAC 1108 VERSION (5-21-68).
           *  *  *  *  *  *  *                                         $
REAL PROCEDURE  ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C1,M)               $
REAL            WN2,DT,C1,M                                            $
INTEGER         HALT,SCRIBE,MULT                                       $
REAL ARRAY      XM,R                                                   $
           BEGIN
REAL            T,WN,ZETA,X,DX,DDX,CF,XO                               $
INTEGER         N,N1,KODE,P                                            $
FORMAT          PFORM1(X9,'TIME',X7,'DISPLACEMENT',X6,'VELOCITY',X6,
                    'ACCELERATION',A1.1),
                PFORM2(X7,D7.4,X6,R12.6,X4,R12.6,X4,R12.6,A0.1)        $
COMMENT    *  *  *  *  *  *  *
           INSERT THE PROCEDURE 'APPROX' HERE'
           *  *  *  *  *  *  *                                         $
COMMENT    *  *  *  *  *  *  *
           THE FOLLOWING PROCEDURE EVALUATES THE EXTENDED K-B METHOD.
           UNDERDAMPED CASE.
           LINEAR, COULOMB, AND VELOCITY SQUARED DAMPING.
           NOTE - INITIAL DISPLACEMENT MUST BE POSITIVE.
           *  *  *  *  *  *  *                                         $
PROCEDURE       APPROX(ZETA,WN,R0,R2,R3,R4,XO,T,XA,DXA,DDXA,KODE,N,N1) $
REAL            ZETA,WN,R0,R2,R3,R4,XO,T,XA,DXA,DDXA                   $
INTEGER         KODE,N,N1                                              $
           BEGIN
REAL            EPSILON,RAD,PHIO,CAPX,C,S,EXPON,SGN,XDOT,PSI,PSIO,SO,CO,
                WD,Z2,TS,EXPONO                                        $
FORMAT          PFORM(X82,'EPSILON = ',R12.6,A1.1)                     $
                EPSILON = R2                                           $
                IF KODE EQL 1 THEN WRITE(PFORM,EPSILON)                $
                KODE = 0                                               $
                Z2 = ZETA*ZETA                                         $
                RAD = SQRT(1.0-Z2)                                     $
                WD = RAD*WN                                            $
                IF ZETA EQL 0.0 THEN
           BEGIN
                PHIO = 3.14159/2.0                                     $
                GO TO PT1                                              $
           END                                                         $
                PHIO = ARCTAN(RAD/ZETA)                                $
           PT1'
                IF N1 EQL 3 THEN GO TO PT3                             $
           PT2'
                IF T LEQ (N*3.14159)/WD THEN
           BEGIN
```

```
      TS = (N-1)*(3.14159/WD)                                      $
      CO = COS((WD*TS)+PHIO)                                       $
      SO = SIN((WD*TS)+PHIO)                                       $
      EXPONO = 1.0                                                 $
      C = COS((WD*T)+PHIO)                                         $
      S = SIN((WD*T)+PHIO)                                         $
      EXPON = EXP(-ZETA*WN*(T-(N-1)*3.14159/WD))                   $
      SGN = (-1)**N                                                $
      IF N1 EQL 0 THEN
BEGIN
      XO = ABS((XO-(RO/(WN**2)))/SO)                               $
      N1 = 1                                                       $
END                                                                $
      PSI = (EXPON/(3*WD*(9-8*Z2)))*(ZETA*WN*(15-11*Z2)*(C**3)
            +9*Z2*WD*(S**3)+3*(3-5*Z2)*WD*S*C*C-3*Z2*ZETA*WN*
            S*S*C+6*(4*Z2-3)*(1-Z2)*ZETA*WN*C+6*WD*(2*Z2-3)*
            (2*Z2-1)*S)                                            $
      PSIO = (EXPONO/(3*WD*(9-8*Z2)))*(ZETA*WN*(15-11*Z2)*
             (CO**3)+9*Z2*WD*(SO**3)+3*(3-5*Z2)*WD*SO*(CO**2)
             -3*Z2*ZETA*WN*(SO*SO)*CO+6*(4*Z2-3)*(1-Z2)*ZETA
             *WN*CO+6*WD*(2*Z2-3)*(2*Z2-1)*SO)                     $
      CAPX = XO/(1.0+EPSILON*XO*SGN*(PSI-PSIO))                    $
      XA = CAPX*EXPON*S-(RO*SGN)/(WN**2)                           $
      XDOT = -ZETA*WN*S+WD*C                                       $
      DXA = CAPX*EXPON*XDOT                                        $
      DDXA = -(2*ZETA*WN*DXA)-(WN*WN*XA)-SGN*(RO+R2*DXA*DXA)       $
      GO TO PT4                                                    $
END                                                                $
      TS = (N-1)*(3.14159/WD)                                      $
      CO = COS((WD*TS)+PHIO)                                       $
      SO = SIN((WD*TS)+PHIO)                                       $
      EXPONO = 1.0                                                 $
      PSIO = (EXPONO/(3*WD*(9-8*Z2)))*(ZETA*WN*(15-11*Z2)*
             (CO**3)+9*Z2*WD*(SO**3)+3*(3-5*Z2)*WD*SO*(CO*CO)
             -3*Z2*ZETA*WN*(SO*SO)*CO+6*(4*Z2-3)*(1-Z2)*ZETA
             *WN*CO+6*WD*(2*Z2-3)*(2*Z2-1)*SO)                     $
      TS = N*3.14159/WD                                            $
      C = COS((WD*TS)+PHIO)                                        $
      S = SIN((WD*TS)+PHIO)                                        $
      EXPON = EXP(-ZETA*WN*3.14159/WD)                             $
      PSI = (EXPON/(3*WD*(9-8*Z2)))*(ZETA*WN*(15-11*Z2)*(C**3)
            +9*Z2*WD*(S**3)+3*(3-5*Z2)*WD*S*C*C-3*Z2*ZETA*WN*
            S*S*C+6*(4*Z2-3)*(1-Z2)*ZETA*WN*C+6*WD*(2*Z2-3)
            *(2*Z2-1)*S)                                           $
      SGN = (-1)**N                                                $
      CAPX = XO/(1.0+EPSILON*XO*SGN*(PSI-PHIO))                    $
      XO = CAPX*EXPON*S-(RO*SGN)/(WN**2)                           $
      IF SIGN(XO) EQL SGN THEN
BEGIN
      IF ABS(XO) GTR RO/(WN*2) THEN
BEGIN
      N = N+1                                                      $
      N1 = 0                                                       $
      XO = ABS(XO)                                                 $
      GO TO PT2                                                    $
END                                                                $
```

```
         END                                                       $ A086
         PT3'                                                        A087
              XA = XO                                              $ A088
              DXA = 0.0                                            $ A089
              DDXA = 0.0                                           $ A090
              N1 = 3                                               $ A091
         PT4'                                                        A092
         END                                                       $ A093
COMMENT  *  *  *  *  *  *  *                                         A094
         THIS IS THE END OF THE PROCEDURE 'APPROX'.                  A095
         *  *  *  *  *  *  *                                       $ A096
              N = 1                                                $ E020
              N1 = 0                                               $ E021
              KODE = 0                                             $ E022
              T = 0.0                                              $ E023
              WN = SQRT(WN2)                                       $ E024
              ZETA = R(1)/(2*WN)                                   $ E025
         PTA1'                                                       E025A
              IF ABS(ZETA) GEQ 1.0 THEN                              E025B
         BEGIN                                                       E025C
              C1 = 0.95*C1                                         $ E025D
              ZETA = C1/(2*M*WN)                                   $ E025E
              R(1) = C1/M                                          $ E025F
              GO TO PTA1                                           $ E025G
         END                                                       $ E025H
              XO = XM(0)                                           $ E026
              X = XO                                               $ E026A
              DX = 0.0                                             $ E027
              DDX = -WN2*X                                         $ E028
              IF SCRIBE EQL 1 THEN                                   E029
         BEGIN                                                       E030
              WRITE(PFORM1)                                        $ E031
              WRITE(PFORM2,T,X,DX,DDX)                             $ E032
         END                                                       $ E033
              P = 0                                                $ E034
              CF = 0.0                                             $ E035
         PT1'                                                        E036
              IF P EQL HALT THEN GO TO PT2                         $ E037
              T = T+DT                                             $ E038
              P = P+1                                              $ E039
              APPROX(ZETA,WN,R(0),R(2),R(3),R(4),XO,T,X,DX,DDX,      E040
                    KODE,N,N1)                                     $ E041
              IF SCRIBE EQL 1 THEN WRITE(PFORM2,T,X,DX,DDX)        $ E042
              CF = CF+((XM(P)-X)**2)                               $ E043
              GO TO PT1                                            $ E044
         PT2'                                                        E045
              ERROR = CF                                           $ E046
         END                                                       $ E047
COMMENT  *  *  *  *  *  *  *                                         E048
         THIS IS THE END OF THE PROCEDURE 'ERROR'.                   E049
         *  *  *  *  *  *  *                                       $ E050
```

NUMERICAL INTEGRATION CRITERION FUNCTION

```
COMMENT     *  *  *  *  *  *  *                                           E001
            NUMERICAL INTEGRATION CRITERION FUNCTION.                     E001A
            THE FOLLOWING PROCEDURE EVALUATES THE LEAST SQUARES           E002
            CRITERION FUNCTION.                                           E003
            UNIVAC 1108 VERSION ( 4-12-68).                               E004
            A DIFFERENTIAL EQUATION OF THE FORM'                          E005
                      M DDX + F(DX) + K X = 0                             E006
            IS INTEGRATED BY THE RUNGE-KUTTA METHOD.                      E007
            F(DX) IS A 4TH ORDER POLYNOMIAL IN THE VELOCITY.              E008
            *  *  *  *  *  *  *                                         $ E009
REAL PROCEDURE  ERROR(WN2,R,DT,HALT,SCRIBE,MULT,XM,C1,M)                $ E010
REAL            WN2,DT,C1,M                                             $ E011
INTEGER         HALT,SCRIBE,MULT                                        $ E012
REAL ARRAY      XM,R                                                    $ E013
            BEGIN                                                         E014
REAL            X,DX,DDX,T,SGN,CF,K1,K2,K3,K4,ADJDX                     $ E015
INTEGER         P,N                                                     $ E016
FORMAT          PFORM1(X9 ,'TIME',X7,'DISPLACEMENT',X6,'VELOCITY',X6,     E017
                     'ACCELERATION',A1.1),                                E018
                PFORM2(X7,D7.4,X6,R12.6,X4,R12.6,X4,R12.6,A0.1)         $ E019
                T = 0.0                                                 $ E020
                X = XM(0)                                               $ E021
                DX = 0.0                                                $ E022
                DDX = -WN2*X                                            $ E023
                IF SCRIBE EQL 1 THEN                                      E024
            BEGIN                                                         E024A
                WRITE(PFORM1)                                           $ E024B
                WRITE(PFORM2,T,X,DX,DDX)                                $ E024C
            END                                                         $ E024D
                P = 0                                                   $ E025
                N = 0                                                   $ E026
                CF = 0.0                                                $ E027
            PT1'                                                          E028
                IF P EQL HALT THEN GO TO PT2                            $ E029
                N = N+1                                                 $ E030
                T = T+DT                                                $ E031
                K1 = DT*DDX                                             $ E032
                ADJDX = DX+0.5*K1                                       $ E033
                SGN = SIGN(ADJDX)                                       $ E034
                K2 = -DT*(WN2*(X+0.5*DT*DX)+R(0)*SGN+ADJDX*(R(1)+         E035
                   ADJDX*(R(2)*SGN+ADJDX*(R(3)+R(4)*SGN*ADJDX))))       $ E036
                ADJDX = DX+0.5*K2                                       $ E037
                SGN = SIGN(ADJDX)                                       $ E038
                K3 = -DT*(WN2*(X+0.5*DT*DX+0.25*DT*K1)+R(0)*SGN+          E039
                   ADJDX*(R(1)+ADJDX*(R(2)*SGN+ADJDX*(R(3)+R(4)*          E040
                   SGN*ADJDX))))                                        $ E041
                ADJDX = DX+K3                                           $ E042
                SGN = SIGN(ADJDX)                                       $ E043
                K4 = -DT*(WN2*(X+DT*DX+0.5*DT*K2)+R(0)*SGN+ADJDX*(R(1)+   E044
```

```
                ADJDX*(R(2)*SGN+ADJDX*(R(3)+R(4)*SGN*ADJDX))))       $
         X = X+DT*DX+(DT*(K1+K2+K3))/6.0                             $
         DX = DX+((K1+2.0*K2+2.0*K3+K4)/6.0)                         $
         SGN = SIGN(DX)                                              $
         DDX = -(WN2*X+R(0)*SGN+DX*(R(1)+DX*(R(2)*SGN+DX*(R(3)+
                R(4)*SGN*DX))))                                      $
         IF SCRIBE EQL 1 THEN WRITE(PFORM2,T,X,DX,DDX)               $
         IF N EQL MULT THEN
    BEGIN
         P = P+1                                                     $
         CF = CF+((XM(P)-X)**2)                                      $
         N = 0                                                       $
    END                                                              $
         GO TO PT1                                                   $
    PT2'
         ERROR = CF                                                  $
    END                                                              $
COMMENT  *  *  *  *  *  *  *
    THIS IS THE END OF THE PROCEDURE ERROR.
    *  *  *  *  *  *  *                                              $
```

APPENDIX E

ADDITIONAL CONSIDERATIONS ON THE APPLICATION OF THE OPTIMUM SEARCH PROCEDURE

Rather than perform the search routine on data obtained for a single initial condition (as is discussed in Chapter V), the search might be performed independently on the data from each of the initial conditions. The unknown system parameters could then be obtained by averaging the damping coefficients resulting from these various runs.

The following discusses this question with respect to the measured data for the subsident system. This additional analysis was conducted on the damping laws corresponding to Cases 3, 4, and 5, Table 3. A comparison of the two techniques is shown in Table 9.

The first column of this table gives the results of performing the search on the response obtained for only one initial condition. Following the search, the damping coefficients were used to predict the response for the other two initial conditions. The least squares criterion function was then calculated for these cases and the sum of the criterion functions for the three initial conditions was determined. The damping law with the lowest sum of criterion functions was then chosen as the best estimate of the unknown coefficients. This is the approach that is shown in Table 4.

With the averaging technique, the search was performed separately on the data obtained for each of the three initial conditions. The average coefficients were determined for each of the damping laws considered. These average coefficients were then used to determine the "averaged" criterion functions. The sum of these, for the three time-histories and various dissipation functions, were obtained and are given in the second column of Table 9. Again, the lowest sum of criterion functions was used to determine the damping law that applied for this system.

Table 9. Comparison of Search Techniques

		SUM OF CRITERION FUNCTIONS	
Case No.*	Damping Law	(Single Search Technique)	(Averaging Technique)
3	C_0+C_1	.2722	.3457
4	C_0+C_2	.2262	.3110
5	$C_0+C_1+C_2$	.5225	.4054

*Corresponds to Tables 3 and 4.

From this table, it is seen that both techniques lead to the same form for the dissipation function, that being Case 4. It is not of significance that the two techniques resulted in different values for the sum of the criterion functions, but rather that both methods lead to the same form for the dissipation function.

The final form of the subsident system's equation of motion was given as follows in Chapter V:

$$0.002137\ddot{x} + (0.03889 + 0.00192\dot{x}^2)\mathrm{Sgn}(\dot{x}) + 0.05042x = 0$$

The equation resulting from the averaging technique is

$$0.002137\ddot{x} + (0.04061 + 0.00160\dot{x}^2)\mathrm{Sgn}(\dot{x}) + 0.05042x = 0$$

The comparison between these two is very acceptable.

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